Recommendation as constrained aggregation

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Motivation

- How to make recommendations to an agent based on information from both:
  - this agent,
  - other agents.
- Applications:
  - Online recommendation systems (collaborative filtering).
  - Experts.
- Normative, axiomatic analysis:
  - Preference aggregation (no strategy).
  - Additional constraint: this agent's information.
  - Missing information.

Contribution

- Formalize the recommendation problem as a constrained aggregation problem.
- State axioms and characterize the unique recommendation rule satisfying them: the "constrained-utilitarian" rule.
- Derive further properties of this rule and compare it with collaborative filtering rules.

Literature (TBA)

- Collaborative filtering (computer science).
  - Normative analysis.
  - Consistency of the recommendation.
- Preference aggregation (social choice).
  - Constrained aggregation.
  - Incomplete preferences.
  - Pin down agents' weights.

Outline

Example I

	a	b	c	d	e
u_0	6	3	4	-	-
u_1	9	5	2	5	8
u_2	3	1	6	5	1

- Agents: $0, 1, 2$. Recommend to 0.
- Alternatives: a, b, c, d, e.
- Ratings on $[0, 10]$ scale: u_0, u_1, u_2.

Example I

	a	b	c	d	e
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u_1	9	5	2	5	8
u_2	3	1	6	5	1

- Agents: $0, 1, 2$. Recommend to 0.
- Alternatives: a, b, c, d, e.
- Ratings on $[0, 10]$ scale: u_0, u_1, u_2.
- Collaborative filter:
  - $\rho(u_0|_{abc}, u_1|_{abc}) \approx 0.71 > 0.22 \approx \rho(u_0|_{abc}, u_2|_{abc})$.
  - $d \mapsto 5$. $e \mapsto 8$ (nearest neighbor) or 6.34 (weighted average).
  - Recommend $e \succ a \succ d \succ c \succ b$.

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    - $d \mapsto 5$. $e \mapsto 8$ (nearest neighbor) or 6.34 (weighted average).
    - Recommend $e \succ a \succ d \succ c \succ b$.
- Constrained utilitarian rule:
    - $u_0|_{abc} = 0.5u_1|_{abc} + 0.5u_2|_{abc}$.
    - $d \mapsto 5$. $e \mapsto 4.5$
    - Recommend $a \succ d \succ e \succ c \succ b$.

Example II

	a	b	c	d
u_0	4	6	-	-
u_1	6	8	5	7
u_2	8	2	5	-

- Agents: $0, 1, 2$. Recommend to 0.
- Alternatives: a, b, c, d.
- Ratings on $[0, 10]$ scale: u_0, u_1, u_2.

Example II

	a	b	c	d
u_0	4	6	-	-
u_1	6	8	5	7
u_2	8	2	5	-

- Agents: $0, 1, 2$. Recommend to 0.
- Alternatives: a, b, c, d.
- Ratings on $[0, 10]$ scale: u_0, u_1, u_2.
- Collaborative filter:
  - $c \mapsto 5$. $d \mapsto 7$.
  - Recommend $d \succ b \succ c \succ a$.

Example II

	a	b	c	d
u_0	4	6	-	-
u_1	6	8	5	7
u_2	8	2	5	-

- Agents: $0, 1, 2$. Recommend to 0.
- Alternatives: a, b, c, d.
- Ratings on $[0, 10]$ scale: u_0, u_1, u_2.
- Collaborative filter:
  - $c \mapsto 5$. $d \mapsto 7$.
  - Recommend $d \succ b \succ c \succ a$.
- Constrained utilitarian rule:
  - $u_0|_{ab} \approx \theta_1 u_1|_{ab} + (1 - \theta_1) u_2|_{ab}$ for $\theta_1 > 0.75$.
  - Recommend $d \succ c$ and $b \succ a \succ c$.

Outline

Alternatives, preferences, utilities

- X: set of alternatives.
  - Non-empty, convex subset of some finite-dimensional vector space.
    - E.g. $X = \Delta(Z)$, Z finite.
- $\succsim$: (weak) preference relation of X.
  - $\succ$: strict preference. $\sim$: indifference.
- $u \in \mathbb{R}^X$: utility function on X.
  - u represents $\succsim$ if $\forall x, y \in X$, $x \succsim y \Leftrightarrow u(x) \geq u(y)$.
- $U \subseteq \mathbb{R}^X$: utility set on X.
  - U represents $\succsim$ if $\forall x, y \in X$, $x \succsim y \Leftrightarrow [\forall u \in U, u(x) \geq u(y)]$.

vNM preorders

- A preference relation $\succsim$ on X satisfies:
  - reflexivity if $\forall x \in X$, $x \succsim x$,
  - completeness if $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$,
  - transitivity if $\forall x, y, z \in X$, $x \succsim y \succsim z \Rightarrow x \succsim z$,
  - mixture continuity if $\forall x, y, z \in X$, the sets $\{\lambda \in [0,1] : x \succsim \lambda y + (1-\lambda)z\}$ and $\{\lambda \in [0,1] : \lambda y + (1-\lambda)z \succsim x\}$ are closed,
  - independence if $\forall x, y, z \in X, \forall \lambda \in (0,1]$, $x \succsim y \Leftrightarrow \lambda x + (1-\lambda)z \succsim \lambda y + (1-\lambda)z$.
- $\succsim$ is a vNM preorder on X if it satisfies reflexivity, transitivity, mixture continuity, and independence.
  - $\mathcal{R}$: set of all vNM preorders on X.

vNM utility sets

- A vNM utility function on X is an affine function $u \in \mathbb{R}^X$.
  - $\forall x, y \in X, \forall \lambda \in [0,1], u(\lambda x + (1-\lambda)y) = \lambda u(x) + (1-\lambda)u(y)$.
  - $\mathcal{U}$: set of all vNM utility functions on X.
- A vNM utility set is a non-empty subset of $\mathcal{U}$.
  - $\mathcal{P}$: set of all vNM utility sets on X.
- A vNM utility set U is simple if $U = \text{conv}(V)$, V finite.
  - $\mathcal{P}_s$: set of all simple vNM utility sets on X.

vNM representation

- A binary relation on X is a vNM preorder if and only if it can be represented by a vNM utility sets (Dubra et al., 2004).
- It is a "finitely generated" vNM preorder if and only if it can be represented by a simple vNM utility set (Dubra and Ok, 2002).
- It is a complete vNM preorder if and only if it can be represented by a vNM utility function (von Neumann and Morgenstern, 1944).
- The vNM utility representation is cardinally unique.

Outline

Recommendation rules

- Agents $0, 1, \dots, I$, I finite.
- Recommend to 0.

Definition

A recommendation rule is a function $F : \mathcal{P}_s^{I+1} \to \mathcal{R}$, $(U_0, U_1, \dots, U_I) \mapsto \succsim_{(U_0, U_1, \dots, U_I)}$, such that for all $(U_0, U_1, \dots, U_I) \in \mathcal{P}_s^{I+1}$ and all $x, y \in X$,

- if $u_0(x) \geq u_0(y)$ for all $u_0 \in U_0$ then $x \succsim_{(U_0, U_1, \dots, U_I)} y$,
- if $u_0(x) > u_0(y)$ for all $u_0 \in U_0$ then $x \succ_{(U_0, U_1, \dots, U_I)} y$.

Compatible profiles

Definition

A profile $(U_0, U_1, \ldots, U_I) \in \mathcal{P}_s^{I+1}$ is compatible if there exist no $x, y \in X$ such that one of the following two conditions holds:

- $u_0(x) > u_0(y)$ for all $u_0 \in U_0$ and $u_i(x) \leq u_i(y)$ for all $u_i \in U_i$ and all $i = 1, \ldots, I$,
- $u_0(x) \geq u_0(y)$ for all $u_0 \in U_0$ and $u_i(x) < u_i(y)$ for all $u_i \in U_i$ and all $i = 1, \ldots, I$.

- $\mathcal{D}_c$ set of all compatible profiles.

Outline

Pareto

Axiom (Pareto)

For all $(U_0, U_1, \ldots, U_I) \in \mathcal{D}_c$ and all $x, y \in X$, if $u_i(x) \geq u_i(y)$ for all $u_i \in U_i$ and all $i \in I$, then $x \succsim_{(U_0, U_1, \ldots, U_I)} y$.

Monotonicity

Axiom (Monotonicity)

For all $(U_0, U_1, \ldots, U_I), (U_0', U_1, \ldots, U_I) \in \mathcal{D}_c$ such that $U_0' \subseteq U_0$ and all $x, y \in X$, if $x \succsim_{(U_0, U_1, \ldots, U_I)} y$ then $x \succsim_{(U_0', U_1, \ldots, U_I)} y$.

Outline

The constrained-utilitarian set

Definition

Given a profile $(U_0, U_1, \ldots, U_I) \in \mathcal{P}_s^{I+1}$, the constraind-utilitarian set $\Omega_{(U_0, U_1, \ldots, U_I)}$ is the set of vNM utility functions $u \in \mathcal{U}$ such that:

- $u = \sum_{i=1}^{I} \theta_i u_i$ for some $\theta \in \Delta(I)$ and $(u_i)_{i=1}^{I} \in \prod_{i=1}^{I} U_i$,
- $u = \alpha u_0 + \beta$ for some $\alpha \in \mathbb{R}_{++}$, $\beta \in \mathbb{R}$, and $u_0 \in U_0$.

Compatible profiles

Proposition

A profile $(U_0, U_1, \ldots, U_I)$ *is compatible if and only if* $\Omega_{(U_0, U_1, \ldots, U_I)}$ *is non-empty.*

The constrained-utilitarian rule

Theorem

- *A recommendation rule F satisfies Pareto if and only if, for all $(U_0, U_1, U_1, \dots, U_I) \in \mathcal{D}_c$, some subset of $\Omega_{(U_0, U_1, \dots, U_I)}$ represents $\succsim_{(U_0, U_1, \dots, U_I)}$.*

The constrained-utilitarian rule

Theorem

- *A recommendation rule F satisfies Pareto if and only if, for all $(U_0, U_1, U_1, \ldots, U_I) \in \mathcal{D}_c$, some subset of $\Omega_{(U_0, U_1, \ldots, U_I)}$ represents $\succsim_{(U_0, U_1, \ldots, U_I)}$.*
- *A recommendation rule F satisfies Pareto and Monotonicity if and only if, for all $(U_0, U_1, U_1, \ldots, U_I) \in \mathcal{D}_c$, $\Omega_{(U_0, U_1, \ldots, U_I)}$ represents $\succsim_{(U_0, U_1, \ldots, U_I)}$.*

Outline

Agents' weights I

Definition

Given a profile $(U_0, U_1, \ldots, U_I) \in \mathcal{P}_s^{I+1}$, the constraind-utilitarian weighting-set $\Theta_{(U_0,U_1,\ldots,U_I)}$ is the set of weight vectors $\theta \in \Delta(I)$ such that:

- $\sum_{i=1}^{I} \theta_i u_i \in \Omega_{(U_0,U_1,\ldots,U_I)}$ for some $(u_i)_{i=1}^{I} \in \prod_{i=1}^{I} U_i$.

Agents' weights II

Proposition

For all $(U_0, U_i, U_j, U_{-0ij}) \in \mathcal{D}_c$ *such that* $U_i = \lambda U_j + (1-\lambda)U_0$ *for some* $\lambda \in (0,1)$*:*

- *For all* $\theta \in \Theta_{(U_0,U_i,U_j,U_{-0ij})}$*, there exists* $\mu \in (0,1)$ *such that* $((1-\mu+\mu\theta_i+\mu\theta_j)_i, 0_j, \mu\theta_{-0ij}) \in \Theta_{(U_0,U_i,U_j,U_{-0ij})}$*.*
- $\sup_{\Theta_{(U_0,U_i,U_j,U_{-0ij})}} \theta_i \geq \sup_{\Theta_{(U_0,U_i,U_j,U_{-0ij})}} \theta_j$*.*
- $\inf_{\Theta_{(U_0,U_i,U_j,U_{-0ij})}} \theta_i \geq \inf_{\Theta_{(U_0,U_i,U_j,U_{-0ij})}} \theta_j = 0$*.*

Agents' weights III

Proposition

For all $(U_0, U_i, U_{-0i}), (U_0, U_i', U_{-0i}) \in \mathcal{D}_c$ *such that* $U_i = \lambda U_i' + (1-\lambda) U_0$ *for some* $\lambda \in (0,1)$*:*

- *For all* $\theta \in \Theta_{(U_0, U_i', U_{-0i})}$, *there exists* $\mu \in (0,1)$ *such that* $((1-\mu+\mu\theta_i)_i, \mu\theta_{-0i}) \in \Theta_{(U_0, U_i, U_{-0i})}$.
- $\sup_{\Theta_{(U_0, U_i, U_{-0i})}} \theta_i \geq \sup_{\Theta_{(U_0, U_i', U_{-0i})}} \theta_i$.
- $\Omega_{(U_0, U_i, U_{-0i})} \supseteq \Omega_{(U_0, U_i', U_{-0i})}$.

Monotonicity

Proposition

For all $(U_0, U_1, \ldots, U_I), (U_0, U_1', \ldots, U_I') \in \mathcal{D}_c$ *such that* $U_i' \subseteq U_i$ *for all* $i = 1, \ldots, I$*:*

- $\Omega_{(U_0, U_1, \ldots, U_I)} \supseteq \Omega_{(U_0, U_1', \ldots, U_I')}$.

Invariance

Proposition

For all $(U_0, U_1, \ldots, U_I), (U_0', U_1', \ldots, U_I') \in \mathcal{D}_c$ *if there exist* $\alpha_0, \alpha \in \mathbb{R}_{++}$ *and* $\beta_0, \beta_1, \ldots, \beta_I \in \mathbb{R}$ *such that* $U_0' = \alpha_0 U_0 + \beta_0$ *and* $U_i' = \alpha U_i + \beta_i$ *for all* $i = 1, \ldots, I$, *then:*

- $\Omega_{(U_0, U_1, \ldots, U_I)} = \Omega_{(U_0, U_1', \ldots, U_I')}$.
- $\Theta_{(U_0, U_1, \ldots, U_I)} = \Theta_{(U_0, U_1', \ldots, U_I')}$.

Conclusion

- Summary:
  - Formalize recommendation as constrained aggregation.
  - Allow for incompleteness.
  - Characterize the constrained-utilitarian rule.
  - Study its properties and compare with collaborative filtering.
- Todo:
  - Literature.
  - Compute constrained-utilitarian set?
  - Allow for negative weights?
  - Characterize refinements?
  - Do away with lotteries? linearity?

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