A behavioral model of individual welfare

Eric DANAN *
EUREQua, Université de Paris 1

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Abstract

I distinguish two different concepts of preferences: *behavioral preferences*, which model an agent's choices, and *cognitive preferences*, which model her tastes. Most economic models only use behavioral preferences, yet conduct welfare analysis, which can result in wrong welfare judgements because of the potential incompleteness of tastes and unobservability of cognitive indifference. On the other hand, introducing cognitive preferences poses a problem according to the behavioral methodology prevailing in economics, since these preferences are unobservable. I propose a model which solves this methodological problem by *deriving* the agent's tastes from her choice behavior. The key feature of this model is a *learning-then-acting* condition linking tastes' incompleteness to preference for flexibility. The results shed new light on the literatures on incomplete preferences and preference for flexibility.

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1 Introduction

Two different concepts of *preferences* have, for a long time, coexisted in economic theory, as can be seen by comparing the two following quotations.

> "Of two acts f and g, it is possible that the person prefers f to g. Loosely speaking, this means that, if he were required to decide between f and g, no other acts being available, he would decide on f. [...] I think it of great importance that preference, and indifference, between f and g be determined, at least in principle, by decisions between acts and not by response to introspective questions." (Savage 1954, p17)
>
> "It is conceivable – and may even in a way be more realistic – to allow for cases where the individual is neither able to state which of two alternatives he prefers, nor that they are equally desirable." (Von Neumann and Morgenstern 1944, p19)

According to the first concept, that the agent prefers an alternative a to an alternative a' means that she *chooses* a over a'; according to the second one, it means that she *desires* a more than a'. Call these two concepts *behavioral preferences* and *cognitive preferences*, respectively. The second quotation emphasizes a fundamental difference between them: cognitive preferences may be incomplete (i.e. may not rank any two alternatives), while behavioral preferences are, by definition, complete.

As noted by Sen (1973), most of economic models, by using a single preference relation (or utility function) per agent for both equilibrium determination and welfare analysis, rely on the implicit assumption that these two types of preferences coincide. This assumption, however, is not reasonable, for when the agent chooses a over a', it rules out the two following situations.

— *incomplete tastes.*[1] she does not know which of a and a' she desires the most,

— *unobservable indifference.* she desires a and a' equally.

These two situations are perfectly conceivable, as it cannot be argued, based on the agent's desires, that she makes a sub-optimal choice; hence behavioral and cognitive preferences should be distinguished. In practice, data about agents' preferences is usually collected by observing their choices, in accordance with the *behavioral* methodology prevailing in economics; thus most of economic models

[1] The word *tastes* is used as a synonymous of desires.

conduct welfare analysis using the agents' behavioral preferences. This can lead a social planner applying the Pareto criterion to errors. For example, consider an economy with two agents $(1, 2)$ and two alternatives (a, a'), in which Agent 1 behaviorally and cognitively strictly prefers a to a', and Agent 2 behaviorally strictly prefers a' to a. Then the social planner considers both a and a' Pareto-optimal, hence might decide to implement a'. Yet it is possible that Agent 2 is cognitively indifferent between a and a', in which case a is the only actual Pareto-optimum, or that Agent 2 is unable to compare a and a', in which case one can argue that a is socially superior to a', since it leaves Agent 1 strictly better off without leaving Agent 2 worse off.

Thus, economic models would better introduce two preference relations per agent: her behavioral preference relation for equilibrium determination and her cognitive preference relation for welfare analysis. This approach, however, poses a methodological problem: tastes, unlike choice behavior, are unobservable, thus any model built on cognitive preferences is not empirically testable. The goal of this paper is to solve this issue. In the remainder of the introduction, I shall first discuss in detail this methodological problem, then present an overview of the proposed solution.

1.1 Methodology

First some terminology. A *model* can be thought of as starting from a set of *primitive concepts*, on which *axioms* are imposed, then logically deducing a set of *results* from these axioms, possibly using a set of *derived concepts*, i.e. concepts that are defined in terms of the primitive ones. In an economic model, a concept does not merely consist in a mathematical object, but is also attached an *interpretation*, i.e. a connection between the mathematical object and some phenomenon described in informal terms. Moreover, a subset of the primitive concepts is always devoted to modelling the set of agents and their environment; call them *structural concepts*, call *structural axiom* any axiom involving structural concepts only, and reserve the name *axiom* for non-structural axioms. Call a concept *behavioral* if it is interpreted in terms of choice behavior, and *cognitive* if it is interpreted in terms of introspection. As far as I know, all economic concepts are either structural, or behavioral, or cognitive. Call a model *behaviorist* if all its concepts are structural or behavioral, *behavioral* if all its primitive concepts are structural or behavioral but at least one of its derived concepts is cognitive, and *cognitive* if at least one of its primitive concepts is cognitive.[2]

A model taking both the agent's behavioral and cognitive preferences as primitive concepts is a cognitive model. Economic theory has spent a great deal of effort in order to avoid using cognitive models, because cognitive concepts, unlike behavioral concepts, are unobservable, hence axioms imposed on them are intestable. Thus, Mas-Colell, Whinston, and Green (1995, p5) introduced the theory of individual decision making as follows.

> "There are two distinct approaches to medelling individual choice behavior. The [*preference-based* approach] treats the agent's tastes, as summarized in her preference relation, as the primitive characteristic of the individual. The theory is developed by first imposing rationality axioms on the agent's preferences and then analyzing the consequences of these preferences for her choice behavior. [...] The [*choice-based* approach] treats the individual's choice behavior as the primitive feature and proceeds by making assumptions directly concerning this behavior. [...] it makes clear that the theory of individual decision making need not be based on a process of introspection but can be given an entirely behavioral foundation."

The preference-based approach is traditionally named *ordinal utility theory*, and traced back to Pareto (1906); it is a cognitive approach. The choice-based approach is traditionally named *revealed preferences theory*, and traced back to Samuelson (1938), whose motivation was, in reaction to ordinal utility theory, "to develop the theory of consumer's behavior freed from any vestigial traces of the utility concept", i.e. a behaviorist model of individual consumption.

As for testability, a behaviorist model is undeniably more desirable than a cognitive one. However, ruling out all cognitive concepts has the major drawback of necessarily leaving aside welfare analysis, a drawback which was also noted by Mas-Colell, Whinston, and Green (1995, p80).

> "Although many of the positive results in consumer theory could also be deduced using [the choice-based approach], the preference-based approach to consumer demand is of critical importance for welfare analy-

[2]This terminology is borrowed from Gilboa and Schmeidler (2001b).

> sis. Without it, we would have no means of evaluating the consumer's level of well-being."

A solution to this weakness of behaviorist models consists in adopting a behavioral viewpoint instead: if agents' tastes can be derived from behavioral primitive concepts, then welfare analysis can take place while all axioms can be empirically tested. In this regard, it is tempting to interpret subsequent developments of revealed preferences theory as achieving the desired goal. Indeed, at least since the issue of "integrability" of demand was addressed (e.g. Houthakker 1950), and then abstracted from the consumption structural setting (e.g. Richter 1966, Sen 1971), revealed preferences theory is generally thought of as deriving the agent's preferences from her "choice function", i.e. her choice behavior in arbitrary choice situations (not only choice between two alternatives).[3] More precisely, taking as primitive concept the agent's choice function, revealed preferences theory derives a preference relation which *rationalizes* it, i.e. whose maximization coincides with the observed choice behavior. So, it is said, the agent behaves *as if* she had such preferences and determined her choice behavior by maximizing them.

Thus, if the preferences derived by revealed preferences theory are interpreted as cognitive preferences, then one has a behavioral model in which they are derived from observable choice behavior. Is this interpretation reasonable? Strictly speaking, the interpretation of these preferences is that they are the preferences whose maximization yields the agent's observed choice behavior. From this, it follows that the agent prefers an alternative a to an alternative a' if and only if she chooses a over a', so these preferences for sure coincide with the agent's behavioral preferences. Hence, interpreting them as the agent's cognitive preferences amounts to assume that tastes and choice behavior always coincide; I shall name this assumption the *transparency* assumption, as it asserts the transparency of the agent's introspection process (her so-called "black box"), in the sense of reducing it to her observable choice behavior. Call *interpretive assumption* any such assumption, i.e. any informal assumption which is only used for interpreting derived concepts.

Clearly, the transparency assumption is not reasonable, as it rules out incomplete tastes and unobservable indifference. But without this assumption, revealed preferences theory falls short of deriving the agent's tastes from her choice behavior: it only derives her binary choice behavior from her choice behavior in more

[3] On this historical evolution of revealed preferences theory, see e.g. Mongin (2000).

general situations. In this paper, I shall propose a behavioral model which I name the *BC-preferences model*, and which completes this behavioral agenda by filling the existing gap between revealed preferences theory and ordinal utility theory. More precisely, I shall take as primitive concept the agent's binary choice behavior, i.e. the ending point of revealed preferences theory, and derive preferences that can reasonably be interpreted as modelling her tastes, i.e. the starting point of ordinal utility theory. This model, of course, is not free from interpretive assumptions, as such assumptions are the price to pay for being able to conduct welfare analysis in a behavioral model. But as I will argue, the interpretive assumptions to be imposed here are more reasonable than the transparency assumption.

1.2 Overview of the results

In order to derive cognitive preferences from behavioral preferences, I shall adopt the methodology of revealed preferences theory. Note that in the derivation of preferences from a primitive choice function, that an alternative a is preferred to an alternative a' if and only if a is chosen over a' is not directly assumed, but follows from the assumption that the derived preferences rationalize the primitive choice function. Thus, the derived concept is not determined by directly applying a constructive definition, but by imposing *conditions* which identify it. A fundamental theorem of revealed preferences theory (Sen 1971, Theorem 9) states that a choice function satisfies some axioms (Properties α and γ) if and only if there exists a binary relation rationalizing it; but a preliminary to this theorem, which is generally omitted because it is trivial, is that two distinct binary relations cannot rationalize the same choice function. Without this preliminary uniqueness result, one would end up without having identified the derived preferences. Similarly here, in order to derive cognitive preferences from behavioral preferences, I shall first solve the *uniqueness problem* (i.e. impose conditions which can be satisfied by at most one cognitive preference relation), and then the *existence problem* (i.e. axiomatize the existence of cognitive preferences satisfying the imposed conditions).[4]

Among the imposed conditions, some may involve the derived concept only, but at least one must link it to the primitive concept (otherwise the derived concept is independent from the primitive one, so there is no "derivation"); call *linking condition* any such condition (for example, the rationalization condition in revealed

[4]Note that this methodology is also the same as that of representing primitive preferences by a utility function.

preferences theory is a linking condition). Here, the first (linking) condition to be imposed asserts *consistency* between the agent's behavioral and cognitive preferences, in the sense that she never chooses an alternative a over an alternative a' while desiring a' strictly more than a. Compared to the assumption that behavioral and cognitive preferences always coincide, the consistency condition allows for incomplete tastes and unobservable indifference, hence establishes a more reasonable link between these two concepts. However, the potentiality of incomplete tastes and unobservable indifference causes non-uniqueness of the derived cognitive preferences, since when the agent is observed choosing a over a', it is possible that she desires a strictly more than a', or that she desires a and a' equally, or that she does not know which of a and a' she desires the most. Hence I shall impose additional conditions to fix these two uniqueness problems. These additional conditions will come at the cost of imposing some structural axioms on the set of alternatives; however, I shall show that any set of alternatives can be *extended* so as to satisfy these structural axioms.

As an intermediate stage, I shall first concentrate on the problem of incomplete tastes by assuming that cognitive indifference is observable. This will be done by strengthening the consistency condition to a *strong consistency* condition (so I shall call *weak consistency* the original version). It then turns out to be sufficient to assume that alternatives are *opportunity sets*, and use the concept of *preference for flexibility*, introduced by Koopmans (1964). More precisely, I shall impose a condition, which I name *learning-then-acting*, asserting that the agent is cognitively unable to rank two alternatives if and only if she is willing to postpone her choice between them (thus hoping to learn about her tastes before she has to act). This condition is the main novelty of this paper, so I shall extensively argue for its sensibility in the body of the paper; for the moment, just note that it merely formalizes the usual intuitive justification of preference for flexibility by "uncertainty about tastes", as can be seen from Kreps's (1979) example of reservation at a restaurant: if the agent has to choose a restaurant to make a reservation for next Monday, then the learning-then-acting condition asserts that she prefers a restaurant proposing both steak and chicken (i.e. to postpone her choice between steak and chicken until the last moment) to a restaurant proposing only steak or only chicken if and only if she does not know which of steak and chicken she will desire the most on Monday.

The consistency and learning-then-acting conditions solve the uniqueness problem in this intermediate stage, i.e. two different cognitive preference relations satisfying the learning-then-acting condition cannot be strongly consistent with the same behavioral preference relation. As for the existence problem, I shall show that given any behavioral preferences, cognitive preferences satisfying the strong consistency and learning-then-acting conditions exist. I shall then consider an intuitive refinement of the learning-then-acting condition, and axiomatize the existence of cognitive preferences satisfying it.

The problem of unobservable indifference can be solved by means of a classical argument consisting in attaching *monetary bonuses* to the alternatives, and detecting cognitive indifference between a and a' by the fact that each of the two alternatives, if augmented by a small monetary bonus, is chosen over the other. In Danan (2002), I formalized this argument in a similar framework, but with the additional condition that cognitive preferences are complete. Here, as a final stage, I shall adopt the weak consistency condition and incorporate this argument about unobservable indifference. This general approach enables to simultaneously solve the (uniqueness and existence) problems of incomplete tastes and unobservable indifference, thereby deriving a fully cognitive concept of preferences from observed choice behavior.

While all the conditions to be imposed in this paper and their intuitive justifications will be discussed, axioms will only be commented by brief paraphrases which help understanding their meaning. This may be surprising, since it is usual in decision theory to comment extensively on the axioms, but it is actually a logical consequence of the explicit distinction between behavioral and cognitive preferences. Indeed, as emphasized by Sen (1993), intuitive justifications of properties of choice behavior always refer to something external to choice behavior. Here, the agent's tastes play the role of this external thing, and since they are explicitly modelled, there is no intuitive justification to give for the imposed axioms.

The results of the intermediate stage of the BC-preferences model have interesting connections with existing literatures. More precisely, since the problem of incomplete tastes is solved by linking preferences' incompleteness to preference for flexibility, these results can be connected to the literature on incomplete preferences, to the literature on preference for flexibility, and to Arlegi and Nieto's (2001) model, which also connects these two literatures. These connections will be formally established in the body of the paper. The reason why connections are established with the intermediate stage of BC-preferences model rather than the final stage is that, as I shall explain, all the literatures to be considered impose continuity axioms under which weak and strong consistency are equivalent, thereby ruling out unobservable indifference. The concepts of behavioral and cognitive preferences were already present in Mandler's (2001) model, only with a different terminology: he respectively called them revealed and psychological preferences, and showed that the usual justifications for standard properties of preferences rely on a confusion between the two concepts. Finally, this paper's title is a reference to Gilboa and Schmeidler's (2001a) paper: noting that "the literature does not seem to offer a convincing justification for substituting revealed preference for [the concept of welfare]", they proposed a cognitive model in which the agent's level of welfare is derived from comparisons between payoffs and aspiration levels; starting from the same observation, I propose a behavioral model in which the agent's cognitive preferences are derived from her observed choice behavior.

The paper is organized as follows. Section 2 introduces the concepts of behavioral and cognitive preferences, as well as the two versions of the consistency condition. Section 3 adopts the strong consistency condition (i.e. dismisses the issue of unobservable indifference, in an intermediate stage) and solves the problem of incomplete tastes. Section 4 formally connects the proposed solution to the aforementioned existing literatures. Section 5 adopts the weak consistency condition and simultaneously solves the problems of incomplete tastes and unobservable indifference; this section contains the most general results of the paper. Section 6 concludes. All proofs appear in the appendix.

2 BC-preference systems

First some mathematical notation. Given a set $\mathcal{S}$, denote by $\mathcal{P}(\mathcal{S})$ the set of nonempty subsets of $\mathcal{S}$, and by $\#\mathcal{S}$ the cardinality of $\mathcal{S}$. Call *binary relation* on a nonempty set $\mathcal{S}$ any subset of $\mathcal{S}\times\mathcal{S}$, and denote by $\mathcal{B}(\mathcal{S})$ the set of binary relations on $\mathcal{S}$. Given $\curvearrowright\in\mathcal{B}(\mathcal{S})$ and $S\in\mathcal{P}(\mathcal{S})$, define $\curvearrowright|_S\in\mathcal{B}(S)$ by $\curvearrowright|_S=\curvearrowright\cap(S\times S)$, i.e. $\curvearrowright|_S$ is the *restriction* of $\curvearrowright$ to S. Given $\curvearrowright\in\mathcal{B}(\mathcal{S})$ and $s,s'\in\mathcal{S}$, let $s\curvearrowright s'$ stand for $(s,s')\in\curvearrowright$, $s\not\curvearrowright s'$ stand for $(s,s')\notin\curvearrowright$, and $s\curvearrowleft s'$ stand for $(s',s)\in\curvearrowright$ (i.e. $\not\curvearrowright$ and $\curvearrowleft$ are respectively the *complement* and the *dual* of $\curvearrowright$), and say that s and s' are $\curvearrowright$*-comparable* if $[s\curvearrowright s'$ or $s\curvearrowleft s']$. Say that $\curvearrowright\in\mathcal{B}(\mathcal{S})$ is

— *reflexive* if $\forall s\in\mathcal{S}$, $s\curvearrowright s$,

— *irreflexive* if $\forall s\in\mathcal{S}$, $s\not\curvearrowright s$,

— *complete* if $\forall s, s' \in \mathcal{S}$, $[s \curvearrowright s' \text{ or } s \curvearrowleft s']$,
— *symmetric* if $\forall s, s' \in \mathcal{S}$, $s \curvearrowright s' \Rightarrow s \curvearrowleft s'$,
— *asymmetric* if $\forall s, s' \in \mathcal{S}$, $s \curvearrowright s' \Rightarrow s \not\curvearrowleft s'$,
— *antisymmetric* if $\forall s, s' \in \mathcal{S}$, $s \curvearrowright s' \curvearrowright s \Rightarrow s = s'$,
— *transitive* if $\forall s, s', s'' \in \mathcal{S}$, $s \curvearrowright s' \curvearrowright s'' \Rightarrow s \curvearrowright s''$.

One can check that asymmetry implies irreflexivity and antisymmetry, and that completeness implies reflexivity. Define $\Theta_{\mathcal{S}}, \Lambda_{\mathcal{S}} \in \mathcal{B}(\mathcal{S})$ by

$$\Theta_{\mathcal{S}} = \mathcal{S} \times \mathcal{S}, \qquad \Lambda_{\mathcal{S}} = \{(s,s) : s \in \mathcal{S}\},$$

i.e. $\Theta_{\mathcal{S}}$ is the maximal (with respect to set inclusion) binary relation on $\mathcal{S}$ and $\Lambda_{\mathcal{S}}$ is the minimal reflexive binary relation on $\mathcal{S}$.

The BC-preferences model's structural concepts are a nonempty set of objectively described and mutually exclusive *alternatives*, denoted by $\mathcal{A}$, and an *agent*. While $\mathcal{A}$ is maintained arbitrary throughout this section, structural axioms will be imposed later on. The non-structural concepts are the agent's *preferences* over $\mathcal{A}$. Preferences over $\mathcal{A}$ are modelled by a binary relation $\succsim$ on $\mathcal{A}$, with $a \succsim a'$ being interpreted as "the agent *weakly prefers* a to a'". Given $\succsim \in \mathcal{B}(\mathcal{A})$, define $\succ, \sim, \bowtie \in \mathcal{B}(\mathcal{A})$ by $\forall a, a' \in \mathcal{A}$,

$$a \succ a' \Leftrightarrow a \succsim a' \not\succsim a, \qquad a \sim a' \Leftrightarrow a \succsim a' \succsim a, \qquad a \bowtie a' \Leftrightarrow a \not\succsim a' \not\succsim a.$$

$a \succ a'$ is interpreted as "the agent *strictly prefers* a to a'", $a \sim a'$ as "the agent *is indifferent* between a and a'", and $a \bowtie a'$ as "the agent *does not rank* a and a'". One can check that

— $\succ$ is asymmetric, $\sim$ and $\bowtie$ are symmetric,
— $\{\prec, \sim, \bowtie, \succ\}$ is a partition of $\mathcal{A} \times \mathcal{A}$,
— $\succsim$ is reflexive if and only if $\sim$ is reflexive, i.e. $\bowtie$ is irreflexive,
— $\succsim$ is complete if and only if $\bowtie = \emptyset$,
— $\succsim$ is antisymmetric if and only if $\forall a, a' \in \mathcal{A}$, $a \sim a' \Rightarrow a = a'$.

A specificity of the BC-preferences model is that the agent is endowed with two different kinds of preferences: *behavioral preferences* and *cognitive preferences*. The binary relations modelling them are respectively denoted by $\succsim_B$ and $\succsim_C$, with $a \succsim_B a'$ being interpreted as "the agent *chooses* a over a'" and $a \succsim_C a'$ as "the agent *desires* a at least as much as a'"; thus, as their names indicate, these two concepts are respectively behavioral and cognitive. Throughout the sequel, I shall restrict attention on complete behavioral preferences and reflexive cognitive preferences, i.e. I shall assume the agent can be forced to choose between any two alternatives and, although she may not cognitively rank any two alternatives, she desires any alternative exactly as much as itself. Hence the following definition.

Definition. *A* ***behavioral preference relation*** *on $\mathcal{A}$ is a complete binary relation on $\mathcal{A}$. A* ***cognitive preference relation*** *on $\mathcal{A}$ is a reflexive binary relation on $\mathcal{A}$. A* ***BC-preference system*** *on $\mathcal{A}$ is a couple $(\succsim_B, \succsim_C) \in \mathcal{B}(\mathcal{A}) \times \mathcal{B}(\mathcal{A})$ such that $\succsim_B$ is a behavioral preference relation and $\succsim_C$ is a cognitive preference relation.*

Call *purely behavioral preference relation* any antisymmetric behavioral preference relation. It seems natural to also restrict attention to purely behavioral preference relations, as choosing means selecting one, and only one alternative. However, this is not the standard approach to modelling choice behavior, as behavioral preferences/choice functions are usually allowed to select several alternatives at once. Thus Savage (1954, p17), immediately after stating his behavioral definition of preferences (quoted in the introduction), commented as follows.

> "This procedure for testing preference is not entirely adequate, if only because it fails to take account of, or even define, the possibility that the person may not really have any preference between f and g, regarding them as equivalent; in which case his choice of f should not be regarded as significant."

He then suggested to define indifference by the fact that any of the two alternatives, if augmented by a small monetary bonus, is chosen over the other.[5] Other authors have proposed different devices meant to observe the agent "regarding two alternatives as equivalent", such as making her choose several times, or offering her the possibility of randomizing between alternatives or of letting another agent choose in her place. These devices are as many attempts to go beyond the behavioral concept of preferences, but they are unsuccessful, if only because they do not take into account the potential incompleteness of tastes. Since the BC-preferences model explicitly treats tastes and choice behavior as two distinct concepts, it is more desirable to stick to the purely behavioral interpretation of $\succsim_B$, according to

[5] This suggestion will be formalized in Section 5.

which the agent selects a single alternative. However, as explained in the introduction, it is useful, in an intermediate stage, to assume that cognitive indifference is observable, and this will be done by means of a *hybrid* interpretation of $a \succsim_B a'$ as "the agent either chooses a over a' or is cognitively indifferent between a and a'", under which antisymmetry of $\succsim_B$ is too restrictive an assumption since it rules out cognitive indifference. Furthermore, even in the final stage of the model (Section 5), it turns out that no formal result relies on $\succsim_B$ being antisymmetric. Hence I shall never impose this axiom, though one should keep in mind that the the BC-preferences model, in its final stage, allows for the purely behavioral interpretation of $\succsim_B$.

In the BC-preferences model, $\succsim_B$ is primitive and $\succsim_C$ derived, thus the model is behavioral. As already mentioned, the interest of a behavioral model, compared to a cognitive one, lies in a descriptive viewpoint: from a normative viewpoint, it is acceptable to treat $\succsim_C$ as a primitive concept too and analyze the "rationality" of $(\succsim_B, \succsim_C)$. From a descriptive viewpoint, it is important not to use such rationality properties (e.g. transitivity) for the derivation of $\succsim_C$, so that rationality can be empirically tested. In order to derive $\succsim_C$, I shall use the methodology, described in the introduction, of imposing conditions and linking conditions. All the conditions and linking conditions to be imposed will be justified by reasonable interpretive assumptions. Of course, the simplest way of deriving $\succsim_C$ from $\succsim_B$ consists in imposing the linking condition $\succsim_C = \succsim_B$, in which case the interpretation of $\succsim_C$ as modelling the agent's tastes amounts to the transparency assumption. But since this latter assumption is not reasonable, a weaker linking condition is needed. The following one, which comes in two versions, will do the job.

Definition. *A BC-preference system* $(\succsim_B, \succsim_C)$ *on* $\mathcal{A}$ *is*

— ***weakly consistent*** *if* $\succ_C \subseteq \succ_B$,

— ***strongly consistent*** *if* $\succ_C \subseteq \succ_B$ *and* $\sim_C \subseteq \sim_B$.

Weak consistency asserts that if the agent cognitively strictly prefers a to a', then she behaviorally strictly prefers a to a'. Strong consistency adds that if she is cognitively indifferent between a and a', then she is behaviorally indifferent between a and a'. Figure 1 illustrates the consistency linking condition, given two alternatives $a, a' \in \mathcal{A}$. The three rectangles (resp. the four ovals) represent all possible states of the agent's behavioral (resp. cognitive) preferences concerning a and a'. The arrows represent all possible states of the agent's BC-preference

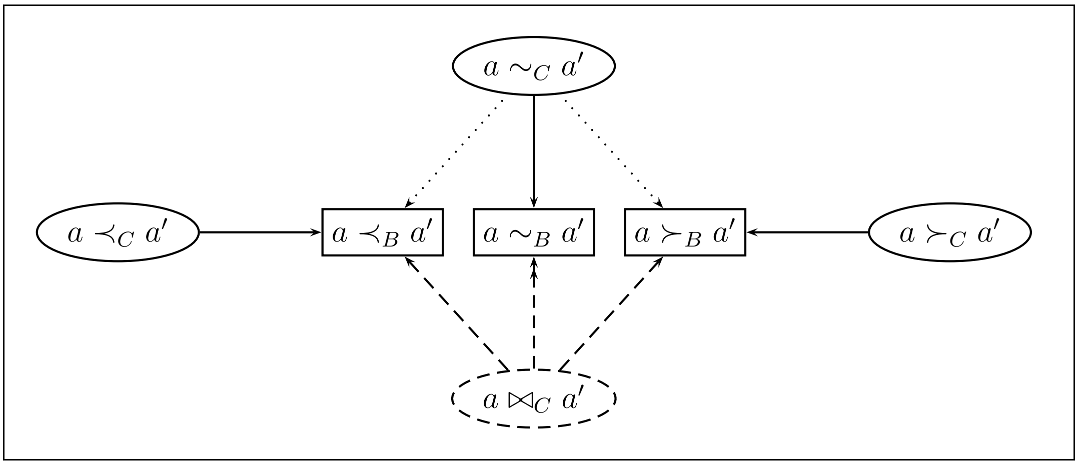

Figure 1: The consistency linking condition

system concerning a and a' which do not violate weak consistency. Under strong consistency, the two dotted arrows disappear. Finally, under $\succsim_C = \succsim_B$ (which is equivalent to the conjunction of strong consistency and completeness of $\succsim_C$), the dashed part of the figure disappears too.[6]

The interpretive assumption underlying the consistency linking condition is that the agent does not act in contradiction with her tastes by choosing an alternative she desires strictly less than another available one (call it the *consistency* assumption); it is weaker, and arguably more reasonable, than the transparency assumption. Under the purely behavioral interpretation of $\succsim_B$, the consistency assumption justifies weak consistency, while strong consistency is too restrictive since cognitive indifference is unobservable. Under the hybrid interpretation of $\succsim_B$, strong consistency obviously becomes sensible. Thus, to each interpretation of $\succsim_B$ corresponds a version of the consistency linking condition. Yet the only formal difference between the two versions is that the latter implies the former, and hence will be useful in an intermediate stage. To understand the interest of this intermediate stage, let us examine how weak (resp. strong) consistency helps solving the uniqueness and existence problems. To make things formal, denote, given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, by $\mathcal{R}_{C^-}(\succsim_B)$ (resp. $\mathcal{R}_{C^+}(\succsim_B)$) the set of cognitive preference relations $\succsim_C$ on $\mathcal{A}$ such that $(\succsim_B, \succsim_C)$ is a weakly (resp. strongly) consistent preference system (hence $\mathcal{R}_{C^+}(\succsim_B) \subseteq \mathcal{R}_{C^-}(\succsim_B)$, obviously).

[6]The meaning of the double-headed arrow will be explained in Section 4.

Let

$$\mathcal{B}_{C^-}(\mathcal{A}) = \{\succsim_B \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{C^-}(\succsim_B) \neq \emptyset\},$$
$$\mathcal{B}_{C^+}(\mathcal{A}) = \{\succsim_B \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{C^+}(\succsim_B) \neq \emptyset\},$$

i.e. $\mathcal{B}_{C^-}(\mathcal{A})$ (resp. $\mathcal{B}_{C^+}(\mathcal{A})$) is the set of behavioral preference relations $\succsim_B$ on $\mathcal{A}$ such that there exists a cognitive preference relation $\succsim_C$ on $\mathcal{A}$ such that $(\succsim_B, \succsim_C)$ is a weakly (resp. strongly) consistent preference system. The uniqueness problem under weak (resp. strong) consistency would be solved if for all $\succsim_B$, $\#\mathcal{R}_{C^-}(\succsim_B) \leq 1$ (resp. $\#\mathcal{R}_{C^+}(\succsim_B) \leq 1$). Unfortunately, this is not the case. Intuitively, if $a \succ_B a'$, then under strong consistency, it is possible that the agent's tastes coincide with her choice behavior (i.e. $a \succ_C a'$), or that they do not rank a and a' (i.e. $a \bowtie_C a'$); under weak consistency, it is also possible that they rank a and a' equally (i.e. $a \sim_C a'$). Formally, one can check that for all $\succsim_B$, $\{\succsim_B, \Lambda_\mathcal{A}\} \subseteq \mathcal{R}_{C^+}(\succsim_B)$, and $\{\succsim_B, \Lambda_\mathcal{A}, \Theta_\mathcal{A}\} \subseteq \mathcal{R}_{C^-}(\succsim_B)$.[7] Thus, there are two sources of non-uniqueness under weak consistency: the potential incompleteness of tastes and the unobservability of cognitive indifference. The interest of strong consistency is that it eliminates the latter source, so that the only difference between $\succsim_B$ and $\succsim_C$ lies in tastes' incompleteness, as the following lemma shows.

Lemma 1. *A BC-preference system $(\succsim_B, \succsim_C)$ on $\mathcal{A}$ is strongly consistent if and only if $\succsim_C = \succsim_B \setminus \bowtie_C$.*

Even under strong consistency, additional conditions and/or linking conditions are needed to solve the uniqueness problem. Besides consistency, there seems to be no other sensible condition or linking condition to impose in this general structural setting where $\mathcal{A}$ is left arbitrary. Hence I shall impose structural axioms under which new conditions can be formally stated and intuitively justified. All these structural axioms are *essentially unrestrictive*, i.e. any set $\dot{\mathcal{A}}$ of alternatives admits an extension $\mathcal{A}$ which satisfies them. Such structural axioms are interesting since they imply no loss of generality, as long as one is willing to extend the set of alternatives. However, since extending the set of alternatives may be problematic (e.g. for an experimenter), attention will be paid to identifying as small as possible extensions.

[7] Hence $\mathcal{B}_{C^-}(\mathcal{A}) = \mathcal{B}_{C^+}(\mathcal{A}) = \mathcal{B}(\mathcal{A})$, which solves the existence problem, but the existence problem is only interesting after the uniqueness problem has been solved.

3 Strong consistency

This section is devoted to the aforementioned intermediate stage of the BC-preferences model: the uniqueness and existence problems are solved under the strong consistency linking condition; hence one should have the hybrid interpretation of $\succsim_B$ in mind. First some mathematical notation. Given two nonempty sets $\mathcal{S}$ and $\mathcal{T}$, denote by $\mathcal{F}(\mathcal{S} \to \mathcal{T})$ the set of functions mapping $\mathcal{S}$ into $\mathcal{T}$. Sometimes $\mathcal{F}(\mathcal{S} \to \mathcal{T})$ will be denoted by $\mathcal{T}^{\mathcal{S}}$, in which case $f \in \mathcal{F}(\mathcal{S} \to \mathcal{T})$ will be denoted by $(f(s))_{s\in\mathcal{S}}$. Call *operator* on a nonempty set $\mathcal{S}$ any function $\bullet \in \mathcal{F}(\mathcal{S} \times \mathcal{S} \to \mathcal{S})$. Say that an operator $\bullet$ on $\mathcal{S}$ is

— *idempotent* if $\forall s \in \mathcal{S}$, $s \bullet s = s$,

— *commutative* if $\forall s, s' \in \mathcal{S}$, $s \bullet s' = s' \bullet s$,

— *associative* if $\forall s, s' \in \mathcal{S}$, $(s \bullet s') \bullet s'' = s \bullet (s' \bullet s'')$.

In this intermediate stage, it turns out to be sufficient to assume that alternatives are *opportunity sets*, the union of two alternatives being itself an alternative. The corresponding structural axiom comes in the two following versions.

Structural Axiom (F: Flexibility). *There exists an idempotent, commutative, and associative operator $\cup_{\mathcal{A}}$ on $\mathcal{A}$.*

Structural Axiom (F'). *There exists a nonempty set $\mathcal{X}$ such that $\mathcal{A}$ is a subset of $\mathcal{P}(\mathcal{X})$ which is closed under finite union.*

Consider first Structural Axiom F' (which is the standard one). $\mathcal{X}$ is called the set of *options*. An opportunity set $a \in \mathcal{A}$ is interpreted as the commitment to choose an option $x \in a$ at some given later date. Thus Structural Axiom F' is interpreted as modelling period 1 alternatives in 2-periods dynamic decision problems. If the agent chooses a singleton $\{x\} \in \mathcal{A}$ in period 1, then she is committed to choose x in period 2, but the option x is not implemented before period 2 (all options are implemented at the same date, no matter the opportunity set out of which they are chosen). Structural Axiom F' is essentially unrestrictive, as any set $\dot{\mathcal{A}}$ can be extended to the set $\mathcal{A} = \mathcal{P}(\dot{\mathcal{A}})$, which obviously satisfies it. This extension merely consists in considering $\dot{\mathcal{A}}$ a set of options and constructing opportunity sets with these options. One can check that the smallest extension which satisfies Structural Axiom F' for all $\dot{\mathcal{A}}$ is the set of nonempty finite subsets of $\dot{\mathcal{A}}$. Clearly, Structural Axiom F is weaker than Structural Axiom F', and hence is essentially unrestrictive too. The interest of this generalization will appear in Section 5. Throughout this section, all comments and interpretations assume Structural Axiom F', while all formal results only assume Structural Axiom F. The following notation, which presupposes Structural Axiom F (this is reminded by the prefix "F-") will be useful. Say that $\curvearrowright \in \mathcal{B}(\mathcal{A})$ is

— *F-complete* if $\forall a, a' \in \mathcal{A}$, $[a \curvearrowright a \cup_{\mathcal{A}} a'$ or $a \curvearrowleft a \cup_{\mathcal{A}} a']$,

— *F-empty* if $\forall a, a' \in \mathcal{A}$, $[a \not\curvearrowright a \cup_{\mathcal{A}} a'$ and $a \not\curvearrowleft a \cup_{\mathcal{A}} a']$.

Clearly, F-completeness is weaker than completeness and stronger than reflexivity, while F-emptiness is stronger than irreflexivity. A larger opportunity set is interpreted as being more *flexible*, as it leaves the agent more possibilities for her period 2 choice, thus F-completeness (resp. F-emptiness) of $\curvearrowright \in \mathcal{B}(\mathcal{A})$ means that two opportunity sets such that one is more flexible than the other are always (resp. never) $\curvearrowright$-comparable. One can check that $\succsim \in \mathcal{B}(\mathcal{A})$ is F-complete if and only if $\bowtie$ is F-empty. Given $\succsim \in \mathcal{B}(\mathcal{A})$, define $\|, \Vdash \in \mathcal{B}(\mathcal{A})$ by $\forall a, a' \in \mathcal{A}$,

$$a \parallel a' \Leftrightarrow a \prec a \cup_{\mathcal{A}} a' \succ a', \qquad\qquad a \Vdash a' \Leftrightarrow a \sim a \cup_{\mathcal{A}} a'.$$

One can check that $\|$ is symmetric and F-empty, and that $\Vdash$ is F-complete if and only if $\succsim$ is reflexive. $a \parallel a'$ is interpreted as "the agent has a *preference for flexibility* at $\{a, a'\}$". This means that she strictly prefers maintaining the flexibility of being able to choose either in a or in a' to being committed to choose in a, as well as to being committed to choose in a'. In other words, the agent has a preference for postponing the choice between (choosing in) a and (choosing in) a'. $a \Vdash a'$ is interpreted as "the agent has an *indifference to flexibility* at (a, a')". This means that she is indifferent between committing to a and maintaining the flexibility of $a \cup_{\mathcal{A}} a'$.

Why would the agent have a preference for flexibility at $\{a, a'\}$? Koopmans (1964) and Kreps (1979) invoked "uncertainty about [period 2] tastes", as in the example of reservation at a restaurant (see the introduction). Arlegi and Nieto (2001, p151) commented as follows.

> "In certain problems of individual choice, the decision maker has only provisional, or not completely made up preferences among the alternatives at her disposal. When this happens, and it is possible to postpone the final choice, it is plausible to think that the agent would like the chance to maintain higher opportunity sets to choose from."

It is this intuition which drives the additional condition to be imposed in this intermediate stage. It comes in the two following versions.

Condition (F-LA: Learning-then-Acting). $\bowtie_C = \|_C$.

Condition (F-DP: Dynamic Programming). $\succsim_C = \Vdash_C$.

One can check that Condition F-LA implies F-completeness of $\succsim_C$. Before commenting on these two conditions, let us explore the relationship between them.

Lemma 2. *Assume $\mathcal{A}$ satisfies Structural Axiom F. Let $\succsim_C$ be a cognitive preference relation on $\mathcal{A}$ satisfying Condition F-DP. Then*

a. $\forall a, a' \in \mathcal{A}$, $a \cup_{\mathcal{A}} a' \succsim_C a$,

b. $\succsim_C$ *satisfies Condition F-LA.*

Thus, Condition F-DP implies Condition F-LA. The reason for stating these two versions is that Condition F-LA turns ut to be formally sufficient to solve the uniqueness problem, but the interpretive assumptions justifying it also imply Condition F-DP. Hence I shall distinguish between a *general* version of the model, using Condition F-LA, and an *intuitive* version using Condition F-DP.

Condition F-LA asserts that the agent has a cognitive preference for flexibility at $\{a, a'\}$ if and only if a and a' are cognitively incomparable. Intuitively, if she is unable to rank a and a' according to her tastes, then she desires to postpone her choice between a and a' (hoping to learn about her tastes before she has to act), while if she is able to rank a and a' according to her tastes, then she does not value the postponement of her choice (since she has nothing more to learn about her tastes). Note that according to this intuition, $\succsim_C$ models the agent's period 2 tastes, as known by her in period 1. Condition F-LA merely formalizes the standard justification for preference for flexibility. Yet there is an alternative to it for modelling this intuition: one might argue that preference for flexibility would better be defined in behavioral terms, and hence impose the linking condition $\bowtie_C = \|_B$ instead of Condition F-LA. Under strong consistency, however, these two modelling possibilities are equivalent, and a similar result holds for Condition F-DP, as the following lemma shows.

Lemma 3. *Assume $\mathcal{A}$ satisfies Structural Axiom F. Let $(\succsim_B, \succsim_C)$ be a strongly consistent BC-preference system on $\mathcal{A}$. Then*

a. if $\succsim_C$ is F-complete, then $\forall a, a' \in \mathcal{A}$, $a \succsim_C a \cup_{\mathcal{A}} a' \Leftrightarrow a \succsim_B a \cup_{\mathcal{A}} a'$ *and* $a \cup_{\mathcal{A}} a' \succsim_C a \Leftrightarrow a \cup_{\mathcal{A}} a' \succsim_B a$,

b. Condition F-LA is equivalent to $\bowtie_C = \|_B$,
c. Condition F-DP is equivalent to $\succsim_C = \Vdash_B$.

An important point to be noted is that for Condition F-LA to be reasonable, it is necessary that choosing an opportunity set a does not induce any change in the objective description of the options $x \in a$. In particular, the agent must not receive any objective information between periods 1 and 2, otherwise she would be likely to have a preference for flexibility at $\{a, a'\}$ even when she is able to rank a and a' according to her tastes (think, for example, of $\mathcal{X}$ being a set of acts, and the agent observing which state of nature occurs between periods 1 and 2). Since it is likely that the agent would try to acquire such objective information if she could, descriptive tests of the BC-preferences model would better be based on experimental data (as the environment can then be controlled so as to make objective information acquisition impossible) rather than "real life" data.

Interpreting preferences satisfying Condition F-LA as modelling the agent's tastes is underlain by the three following interpretive assumptions.

— *intrinsic comparability.* if the agent is unable to rank two alternatives according to her tastes, then she does not consider them as intrinsically incomparable, in the sense that there is no chance that introspecting her tastes would enable her to learn anything,
— *spontaneous introspection.* if there is something to learn about her tastes, then she is willing to try to learn it by introspection,
— *optionalism.* she desires flexibility only insofar as it allows her to introspect her tastes before having to choose an option.

I shall now argue that these three interpretive assumptions are reasonable in most economic settings. The intrinsic comparability assumption might be violated when there are alternatives involving death or other phenomena that the agent is likely to consider outside the scope of her tastes. Most of economic alternatives, however, do not involve such phenomena. The spontaneous introspection assumption prevents the agent from definitely stopping her introspection process in period 1 because she anticipates that carrying it forward after period 1 is not worth the psychological cost. Such a judgement would be that of an agent who considers introspection as a complex process whose psychological cost can be measured *ex ante*. But this measurement itself induces a psychological cost that must be measured, and so on through an infinite regress. I think it more natural to consider introspection as the process of conditioning oneself to receive signals about oneself's tastes. This process is likely to be costly, as one seeks to perceive decreasingly intense signals, but it is always conceivable that the signal's intensity increases before period 2 (especially if the options are to be implemented shortly after period 2, as in the restaurant example), so that it becomes perceptible at a negligible psychological cost. The optionalism assumption is very similar to the consequentialism assumption in the setting of lotteries; it prevents the agent from valuing (positively or negatively) flexibility *per se*. As far as I know, only two phenomena that can invalidate it have been analyzed in the literature: "intrinsic value of freedom of choice" (e.g. Sen 1988) and "temptation" (e.g. Gul and Pesendorfer 2001). Yet a great deal of economic analysis is not concerned with these two phenomena.

Condition F-DP asserts that the agent desires a at least as much as a' if and only if the flexibility gained by adjoining a' to a is of null cognitive value. Intuitively, the agent, when having to choose between a and a', introspects her tastes concerning the options in $a \cup a'$, and she desires a at least as much as a' if and only if she knows she will desire some option in a at least as much as any option in a' (note that she need not know which option in a she will desire more than a given option in a', and her ranking of options may depend on the pair $\{a, a'\}$ she is considering). Like Condition F-LA, Condition F-DP is intuitively justified by the intrinsic comparability, spontaneous introspection, and optionalism assumptions.

Condition F-LA turns out to be sufficient to solve the uniqueness problem under strong consistency (and hence Condition F-DP is sufficient too), as the following theorem shows.

Theorem 1. *Assume $\mathcal{A}$ satisfies Structural Axiom F. Let $\succsim_B$ be a behavioral preference relation on $\mathcal{A}$, and $\succsim_{C_1}$ and $\succsim_{C_2}$ be two cognitive preference relations on $\mathcal{A}$ satisfying Condition F-LA such that $\forall j \in \{1, 2\}$, $(\succsim_B, \succsim_{C_j})$ is a strongly consistent BC-preference system. Then $\succsim_{C_1} = \succsim_{C_2}$.*

Now for the existence problem. Given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, denote by $\mathcal{R}_{C^+,F^-}(\succsim_B)$ (resp. $\mathcal{R}_{C^+,F^+}(\succsim_B)$) the set of $\succsim_C \in \mathcal{R}_{C^+}(\succsim_B)$ satisfying Condition F-LA (resp. F-DP). Let

$$\mathcal{B}_{C^+,F^-}(\mathcal{A}) = \{\succsim_B \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{C^+,F^-}(\succsim_B) \neq \emptyset\},$$
$$\mathcal{B}_{C^+,F^+}(\mathcal{A}) = \{\succsim_B \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{C^+,F^+}(\succsim_B) \neq \emptyset\},$$

By Lemma 2b, $\mathcal{R}_{C^+,F^+}(\succsim_B) \subseteq \mathcal{R}_{C^+,F^-}(\succsim_B)$, hence $\mathcal{B}_{C^+,F^+}(\mathcal{A}) \subseteq \mathcal{B}_{C^+,F^-}(\mathcal{A})$.

Theorem 2. *Assume $\mathcal{A}$ satisfies Structural Axiom F. Let $\succsim_B$ be a behavioral preference relation on $\mathcal{A}$. Then there exists a cognitive preference relation $\succsim_C$ on $\mathcal{A}$ satisfying Condition F-LA such that $(\succsim_B, \succsim_C)$ is a strongly consistent BC-preference system.*

Thus, in the general version of the model under strong consistency, the existence problem is solved without imposing any axiom on $\succsim_B$. In the intuitive version, the existence problem turns out to be solved by the two following axioms.

Axiom (F-M: Monotonicity). *$\forall a, a' \in \mathcal{A}$, $a \cup_{\mathcal{A}} a' \succsim_B a$.*

Axiom (F-T: Transitivity). *$\forall a, a' \in \mathcal{A}$, $\succsim_B|_{\{a, a', a \cup_{\mathcal{A}} a'\}}$ is transitive.*

Axiom F-M asserts that the agent always chooses an opportunity set over a less flexible one. Axiom F-T is much weaker than transitivity of $\succsim_B$, as it only requires that $a \succsim_B a' \succsim_B a'' \Rightarrow a \succsim_B a''$ when $a = a' \cup_{\mathcal{A}} a''$ or $a' = a \cup_{\mathcal{A}} a''$ or $a'' = a \cup_{\mathcal{A}} a'$.

Theorem 3. *Assume $\mathcal{A}$ satisfies Structural Axiom F. Let $\succsim_B$ be a behavioral preference relation on $\mathcal{A}$. Then $\succsim_B$ satisfies Axioms F-M and F-T if and only if there exists a cognitive preference relation $\succsim_C$ on $\mathcal{A}$ satisfying Condition F-DP such that $(\succsim_B, \succsim_C)$ is a strongly consistent BC-preference system.*

This section's results can be summarized as follows. Concerning the uniqueness problem, given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, $\#\mathcal{R}_{C^+,F^-}(\succsim_B) \leq 1$ by Theorem 1, hence $\#\mathcal{R}_{C^+,F^+}(\succsim_B) \leq 1$ by Lemma 2b. As for the existence problem, $\mathcal{B}_{C^+,F^-}(\mathcal{A}) = \mathcal{B}(\mathcal{A})$ by Theorem 2, and $\mathcal{B}_{C^+,F^+}(\mathcal{A})$ is the set of behavioral preference relation on $\mathcal{A}$ satisfying Axioms F-M and F-T by Theorem 3. In the intuitive version of the model, any additional condition imposed on $\succsim_C$ can easily be characterized in terms of a corresponding axiom imposed on $\succsim_B$. Indeed, $\succsim_C = \Vdash_B$ by Lemma 3c, hence, for instance, $\succsim_C$ is transitive if and only if $\Vdash_B$ is transitive. In the general version of the model, $\succsim_C = \succsim_B \setminus \|_B$ may differ from $\Vdash_B$: for example, if $a \sim_B a \cup a' \sim_B a' \succ_B a$, then $a \Vdash_B a' \succ_C a$. The reason for this discrepancy is that in the general version, if the agent does not have a behavioral preference for flexibility at $\{a, a'\}$, then the state of her cognitive preferences concerning a and a' is derived from her choice between a and a', not between a and $a \cup_{\mathcal{A}} a'$. Axioms F-M and F-T ensure that these two ways of deriving the agent's cognitive preferences are equivalent.

4 Connections with existing literatures

Section 3 has considered the intermediate stage of the BC-preferences model, in which the uniqueness problem lies solely in the potential incompleteness of cognitive preferences, and solved this problem by linking this incompleteness to cognitive/behavioral preference for flexibility. In this section, I shall connect the results of Section 3 to the literature on incomplete preferences, to the literature on preference for flexibility, and to Arlegi and Nieto's (2001) model, which is the only existing analysis of the relationship between these two literatures, as far as I know. For the sake of coherence, I shall translate the terminology and notation of the presented literatures into my own ones. First some mathematical notation. Denote respectively by $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_+^*$ the sets of real numbers, non-negative real numbers, and positive real numbers, and by $\geq$ their natural order. Given a finite set $\mathcal{S}$, let

$$\Delta(\mathcal{S}) = \{\mu \in \mathcal{F}(\mathcal{S} \to \mathbb{R}_+) : \sum\nolimits_{s \in \mathcal{S}} \mu(s) = 1\},$$

i.e. $\Delta(\mathcal{S})$ is the set of *probability distributions* on $\mathcal{S}$.

4.1 Arlegi and Nieto (2001)

Like the BC-preferences model under strong consistency, Arlegi and Nieto's (2001, Theorem 4) model (henceforth *AN model*) takes as primitive a preference relation, from which another, potentially incomplete, preference relation is derived, by means of a condition relating preferences' incompleteness to preference for flexibility. However, the motivation for the AN model lies in the connection between incomplete preferences and preference for flexibility *per se*, not in the behavioral and cognitive interpretations of preferences. Hence the primitive preference relation of the AN model is not necessarily complete, and is simply interpreted as modelling the agent's "preferences", while the derived preference relation is interpreted as modelling "the known portion of her preferences". In order to compare the BC-preferences model to the AN model, I shall assume that these interpretations are respectively behavioral and cognitive, hence that the primitive preference relation is complete. This assumption is plausible, since Arlegi and Nieto (2001, p158) mentioned an analogy between their model and revealed preferences theory. Formally, the AN model imposes the following structural axiom (which is clearly stronger than Structural Axiom F') and linking condition.

Structural Axiom ($\mathbf{F}^{AN}$). *There exists a nonempty, finite set $\mathcal{X}$ such that $\mathcal{A} = \mathcal{P}(\mathcal{X})$.*

Definition. *Assume $\mathcal{A}$ satisfies Structural Axiom (F^{AN}). A couple $(\succsim_B, \dot{\succ}_C) \in \mathcal{B}(\mathcal{A}) \times \mathcal{B}(\mathcal{X})$ such that $\dot{\succ}_C$ is asymmetric is* ***AN-consistent*** *if $\forall a \in \mathcal{A}$, $\forall x' \in \mathcal{X} \setminus \{a\}$,*

$$\begin{cases} a \cup \{x'\} \sim_B a & \text{if } \exists x \in \mathcal{A} \text{ such that } x \dot{\succ}_C x', \\ a \cup \{x'\} \succ_B a & \text{otherwise.} \end{cases}$$

One can check that AN-consistency implies $x \dot{\succ}_C x' \Leftrightarrow [x \neq x'$ and $\{x\} \sim_B \{x, x'\}]$, and hence solves the uniqueness problem. AN-consistency has the same intuitive flavor as the linking condition $\succsim_C = \Vdash_B$ (which is equivalent to Condition F-DP under strong consistency): when having to choose between a and $a \cup \{x'\}$, the agent introspects her tastes concerning the options in $a \cup \{x'\}$, and she chooses a over $a \cup \{x'\}$ if and only if she knows she will desire some option in a strictly more than x' (only now she knows which option $x \in a$ she will desire more than x', and her ranking of options does not depend on the pair $\{a, \{x'\}\}$ she is considering). However, cognitive preferences are only defined over the set of options, and no concept of cognitive indifference is derived.

The AN model assumes a stronger structural axiom than the BC-preferences model under strong consistency. Even if this structural axiom is satisfied, the AN model is less general than the general version of the BC-preferences model, in the sense of imposing stronger axioms (since $\mathcal{B}_{C^+, F^-}\mathcal{A} = \mathcal{B}(\mathcal{A})$, this simply means that the AN model imposes some axioms). If these axioms are satisfied, both the cognitive preference relation $\succsim_C$ on $\mathcal{A}$ derived in the general version of the BC-preferences model and the cognitive strict preference relation $\dot{\succ}_C$ on $\mathcal{X}$ derived in the AN model are well-defined, hence the question arises as to whether they coincide, in the sense that $\forall x, x' \in \mathcal{X}$, $x \dot{\succ}_C x' \Leftrightarrow \{x\} \succ_C \{x'\}$. The answer is negative since, as noted in Section 3, if the agent does not have a behavioral preference for flexibility, then the general version of the BC-preferences model identifies $\{x\} \succ_C \{x'\}$ with $\{x\} \succ_B \{x'\}$, while the AN model identifies $x \dot{\succ}_C x'$ with $\{x\} \sim_B \{x, x'\}$. For example, if $\{x\} \sim_B \{x, x'\} \succ_B \{x'\} \succ_B \{x\}$, then $x \dot{\succ}_C x'$ and $\{x'\} \succ_C \{x\}$.

This discrepancy disappears in the intuitive version of the BC-preferences model, as one can check that if $\succsim_B$ satisfies Axioms F-M and F-T, as well as the axioms of the AN model, then $x \dot{\succ}_C x' \Leftrightarrow \{x\} \succ_C \{x'\}$. However, the AN model is neither more nor less general than the intuitive version of the BC-preferences model: on the one hand, AN-consistency does not allow for cognitive indifference, nor for the agent to know she will desire some $x \in a$ strictly more than $x' \in \mathcal{A} \setminus a$ without knowing which one, nor from her cognitive ranking of options to depend on the pair $\{a, \{x'\}\}$ she is considering; on the other hand, AN consistency is only concerned with the flexibility gained by adjoining a singleton to an opportunity set (hence it implies the axiom $a \cup \{x'\} \succsim_B a$, but not Axiom F-M), and the AN model does not incorporate any consistency linking condition (hence Axiom F-T is not necessary).

4.2 Incomplete preferences

The literature on incomplete preferences typically takes as primitive concept an incomplete preference relation $\succsim$ on $\mathcal{A}$, then axiomatizes the existence of some representation of $\succsim$. A major challenge to this literature was to generalize the traditional concept of representation by a utility function, which prevents preferences from being incomplete; it was overcome by the introduction of the following generalization of the traditional concept.

Definition. *A **utility representation** of $\succsim \in \mathcal{B}(\mathcal{A})$ is a function $u \in \mathcal{F}(\mathcal{A} \to \mathbb{R})$ such that $\forall a, a' \in \mathcal{A}$,*

$$a \succsim a' \Leftrightarrow u(a) \geq u(a').$$

*A **multi-utility representation** of $\succsim \in \mathcal{B}(\mathcal{A})$ is a set $U \in \mathcal{P}(\mathcal{F}(\mathcal{A} \to \mathbb{R}))$ such that $\forall a, a' \in \mathcal{A}$,*

$$a \succsim a' \Leftrightarrow [\forall u \in U,\ u(a) \geq u(a')].$$

The interpretation of a multi-utility representation is that the agent envisions several possible (subjective) “scenarios” affecting her preferences, each of whom yielding a utility function u (hence a complete ranking of alternatives), and prefers a to a' if and only if a has higher utility than a' according to each scenario. Given $\succsim \in \mathcal{B}(\mathcal{A})$, denote by $\mathcal{R}_U(\succsim)$ (resp. $\mathcal{R}_{MU}(\succsim)$) the set of utility (resp. multi-utility) representations of $\succsim$. Let

$$\mathcal{B}_U(\mathcal{A}) = \{\succsim \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_U(\succsim) \neq \emptyset\},$$
$$\mathcal{B}_{MU}(\mathcal{A}) = \{\succsim \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{MU}(\succsim) \neq \emptyset\},$$

Clearly, given $\succsim \in \mathcal{B}_U(\mathcal{A})$ and $u \in \mathcal{R}_U(\succsim)$, $\{u\} \in \mathcal{R}_{MU}(\succsim)$. Hence $\mathcal{B}_U(\mathcal{A}) \subseteq \mathcal{B}_{MU}(\mathcal{A})$. $\mathcal{B}_U(\mathcal{A})$ was first axiomatized by Cantor (1895), and is contained in the set of behavioral preference relations on $\mathcal{A}$. $\mathcal{B}_{MU}(\mathcal{A})$ was axiomatized under various structural axioms[8] (e.g. Bewley 1986, Dubra, Maccheroni, and Ok 2001) and is contained in the set of cognitive preference relations on $\mathcal{A}$.

It is natural to interpret $\succsim$ cognitively, as the literature interprets $\bowtie$ as modelling the agent's inability to rank alternatives. Under this interpretation, the literature on incomplete preferences is clearly cognitive. The BC-preferences model then allows to interpret $\succsim$ as the cognitive preference relation $\succsim_C$ derived from the agent's behavioral preference relation $\succsim_B$ on $\mathcal{A}$ (recall that Structural Axiom F' is essentially unrestrictive), thus overcoming the traditional methodological critique addressed to this theory: that it takes as primitive an unobservable concept. This yields a behavioral model in which the axioms of incomplete preferences theory become conditions which, by Theorem 3, can be simply characterized in terms of axioms imposed on $\Vdash_B$.

As far as I know, the only existing attempt to provide incomplete preferences theory with behavioral foundations is Eliaz and Ok's (2003) model (henceforth *EO model*). Its primitive concept is a choice function on $\mathcal{A}$, but since, as one can check, the imposed axioms imply Sen's (1971) Properties α and γ, it is equivalent to start from a behavioral preference relation $\succsim_B$ on $\mathcal{A}$. The EO model imposes the following linking condition.

Definition. *A BC-preference system $(\succsim_B, \succsim_C)$ on $\mathcal{A}$ is* ***EO-consistent*** *if $\succsim_B = \not\prec_C$.*

EO-consistency asserts that the agent chooses an alternative if and only if she does not desire it strictly less than another available one (clearly, this only makes sense under the hybrid interpretation of $\succsim_B$). One can check that it is stronger than strong consistency. In Figure 1, starting from strong consistency, EO-consistency is represented by dropping the two dashed, single-headed arrows and conserving only the double-headed one. Hence the uniqueness problem is brought down to that of disentangling cognitive incomparability from cognitive indifference (formally, one can check that for all $\succsim_B$, both $(\succsim_B, \succsim_B)$ and $(\succsim_B, \Lambda_{\mathcal{A}} \cup \succ_B)$ are EO-consistent BC-preference systems). A first result (Eliaz and Ok 2003, Theorem 2) shows that $\succ_B$ is transitive if and only if there exists a transitive $\succsim_C$ such that $(\succsim_B, \succsim_C)$ is

[8] Ok (2002) left $\mathcal{A}$ arbitrary, but only identified sufficient axioms.

EO-consistent, but this does not solve the uniqueness problem (if $\succsim_B$ is transitive, then the two latter solutions still work). They then add the structural axiom that $\mathcal{A}$ is a metric space, as well as a "regularity" condition. Under this structural axiom, given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, denote by $\mathcal{R}_{EO}(\succsim_B)$ the set of regular cognitive preference relations $\succsim_C$ on $\mathcal{A}$ such that $(\succsim_B, \succsim_C)$ is EO-consistent. Let

$$\mathcal{B}_{EO}(\mathcal{A}) = \{\succsim_B \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{EO}(\succsim_B) \neq \emptyset\},$$

Clearly, for all $\succsim_B$, $\mathcal{R}_{EO}(\succsim_B) \subseteq \mathcal{R}_{C^+}(\succsim_B)$, hence $\mathcal{B}_{EO}(\mathcal{A}) \subseteq \mathcal{B}_{C^+}(\mathcal{A})$. A second result (Eliaz and Ok 2003, Theorem 3) shows that for all $\succsim_B$, $\#\mathcal{R}_{EO}(\succsim_B) \leq 1$, and that $\mathcal{B}_{EO}(\mathcal{A})$ is the set of $\succsim_B$ satisfying a continuity axiom such that $\succ_B$ is transitive.

The BC-preferences model and the EO model assume two independent structural axioms; hence in order to compare them, one has to impose both axioms simultaneously (e.g. $\mathcal{A}$ is the set of nonempty, compact subsets of a metric space $\mathcal{X}$, endowed with the Hausdorff metric). In this case, the general version of the BC-preferences model is clearly more general than the EO model (i.e. $\mathcal{B}_{EO}(\mathcal{A}) \subset \mathcal{B}_{C^+,F^-}(\mathcal{A})$), since $\mathcal{B}_{C^+,F^-}(\mathcal{A}) = \mathcal{B}(\mathcal{A}) \neq \mathcal{B}_{EO}(\mathcal{A})$. The intuitive version is neither more general nor less general than the EO model, since Axioms F-M and F-T explicitly use the flexibility structural setting, while the EO model's continuity axiom explicitly uses the metric space structural setting.

If the cognitive preference relations derived in the BC-preferences model and in the EO model are both well-defined, then they do not necessarily coincide. Indeed, both versions of the BC-preferences model identify cognitive incomparability with behavioral preference for flexibility, while the EO model identifies cognitive incomparability with intransitive behavioral indifference. For example, if $\succsim_B \in \mathcal{B}_{C^+,F^+}(\mathcal{A}) \cap \mathcal{B}_{EO}(\mathcal{A})$ and $a, a', a'' \in \mathcal{A}$ are such that $a \sim_B a \cup_{\mathcal{A}} a' \sim_B a' \sim_B a \sim_B a'' \succ_B a$, then for all $\succsim_C$, $\succsim_C \in \mathcal{R}_{C^+,F^+}(\succsim_B) \Rightarrow a \sim_C a'$ and $\succsim_C \in \mathcal{R}_{EO}(\succsim_B) \Rightarrow a \bowtie_C a'$. A drawback of the EO model is that it identifies incomplete preferences with irrationality, in the sense that the agent's derived cognitive preferences are incomplete if and only if her behavioral preferences are vulnerable to a "money pump" (unless one deviates from the standard money pump argument by introducing a structural concept of "status quo", e.g. Burros 1974). At the same time, the EO model assumes transitivity of the agent's cognitive preferences, so it follows from Szpilrajn's (1930) Theorem that there exists a rational behavior (i.e. transitive behavioral preferences) which does not contradict her tastes. The BC-preferences model, on the contrary, does not prevent an agent whose tastes are incomplete from taking advantage of such an opportunity to adopt a rational behavior which does not contradict her tastes.

4.3 Preference for flexibility

Within the flexibility structural setting, Kreps (1979) imposed the following structural axiom (which is clearly stronger than Structural Axiom F') and, taking as primitive concept a preference relation $\succsim$ on $\mathcal{A}$, axiomatized the existence of the following type of representation of $\succsim$.

Structural Axiom (F^K). *There exists a nonempty set $\mathcal{X}$ such that $\mathcal{A} = \mathcal{P}(\mathcal{X})$.*

Definition. *Assume $\mathcal{A}$ satisfies Structural Axiom (F^K). A **subjective state space representation** of $\succsim \in \mathcal{B}(\mathcal{A})$ is a couple $(\dot{U}, g)$ such that $\dot{U} \in \mathcal{P}(\mathcal{F}(\mathcal{X} \to \mathbb{R}))$, $g \in \mathcal{F}(\mathbb{R}^{\dot{U}} \to \mathbb{R})$ is strictly increasing, and $u_{\dot{U},g} \in \mathcal{R}_U(\succsim)$, where $u_{\dot{U},g} \in \mathcal{F}(\mathcal{A} \to \mathbb{R})$ is defined by $\forall a \in \mathcal{A}$,*

$$u_{\dot{U},g}(a) = g((\sup_{x \in a} \dot{u}(x))_{\dot{u} \in \dot{U}}).$$

The interpretation of a subjective state space representation is that the agent, in period 1 envisions several possible (subjective) scenarios affecting her period 2 preferences over options, each of whom yielding a utility function $\dot{u}$ (hence a complete ranking of options), anticipates that she will learn which scenario occurs before period 2 (hence her evaluation of $a \in \mathcal{A}$ if scenario $\dot{u}$ occurs will be $\sup_{x \in a} \dot{u}(x)$), and aggregates her period 1 subjective uncertainty about scenarios by means of g. Given $\dot{U} \in \mathcal{P}(\mathcal{F}(\mathcal{X} \to \mathbb{R}))$, call *subjective state space* the set

$$\dot{P}(\dot{U}) = \{\dot{\succsim} \in \mathcal{B}(\mathcal{X}) : \mathcal{R}_U(\dot{\succsim}) \cap \dot{U} \neq \emptyset\},$$

i.e. the set of binary relations on $\mathcal{A}$ of whose some element of $\dot{U}$ is a utility representation. Note that given a subjective state space representation $(\dot{U}, g)$ of $\succsim \in \mathcal{B}(\mathcal{A})$, $\| \neq \emptyset \Leftrightarrow \#\dot{P}(\dot{U}) > 1$; thus, preference for flexibility is represented by subjective uncertainty. Given $\dot{U} \in \mathcal{P}(\mathcal{F}(\mathcal{X} \to \mathbb{R}))$ and $g \in \mathcal{F}(\mathbb{R}^{\dot{U}} \to \mathbb{R})$, say that $\dot{U}$ is *relevant* if $\forall \dot{u} \in \dot{U}$, $\exists a, a' \in \mathcal{A}$ such that $[u_{\dot{U},g}(a) \neq u_{\dot{U},g}(a')$ and $\forall \dot{u}' \in \dot{U} \setminus \{\dot{u}\}$, $\sup_{x \in a} \dot{u}'(x) = \sup_{x' \in a'} \dot{u}'(x')]$, i.e if each $\dot{u} \in \dot{U}$ is "key" for some comparison of opportunity sets by means of $u_{\dot{U},g}$.

Given $\succsim \in \mathcal{B}(\mathcal{A})$, denote by $\mathcal{R}_{SSS}(\succsim)$ the set of subjective state space representations of $\succsim$. Let

$$\mathcal{B}_{SSS}(\mathcal{A}) = \{\succsim \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{SSS}(\succsim) \neq \emptyset\},$$

Clearly, given $\succsim \in \mathcal{B}(\mathcal{A})$ and $(\dot{U}, g) \in \mathcal{R}_{SSS}(\succsim)$, $u_{\dot{U},g} \in \mathcal{R}_U(\succsim)$. Hence $\mathcal{B}_{SSS}(\mathcal{A}) \subseteq \mathcal{B}_U(\mathcal{A})$. It follows from Kreps (1979, Theorem 4) that $\mathcal{B}_{SSS}(\mathcal{A})$ is contained in the set of transitive behavioral preference relations satisfying Axiom F-M. Hence $\mathcal{B}_{SSS}(\mathcal{A}) \subset \mathcal{B}_{C^+,F^+}(\mathcal{A})$.

A shortcoming of Kreps's (1979) model is the lack of uniqueness result for $(\dot{U}, g)$. This was remedied by Dekel, Lipman, and Rustichini (2001), under the following structural axiom.

Structural Axiom (F^{DLR}). *There exists a nonempty, finite set $\mathcal{Z}$ such that $\mathcal{A} = \mathcal{P}(\Delta(\mathcal{Z}))$.*

Compared to Structural Axiom F^K, Structural Axiom F^{DLR} adds that the options are lotteries over a finite set $\mathcal{Z}$ of prizes. This additional structure allows to strengthen the representation concept. First, some mathematical notation is needed. Given a finite set $\mathcal{Z}$, call *$\mathcal{Z}$-expected utility function* any function $\dot{u} \in \mathcal{F}(\Delta(\mathcal{Z}) \to \mathbb{R})$ such that $\exists \ddot{v} \in \mathcal{F}(\mathcal{Z} \to \mathbb{R})$ such that $\forall x \in \Delta(\mathcal{Z})$, $\dot{u}(x) = \sum_{z \in \mathcal{Z}} x(z)\ddot{v}(z)$, and call $\ddot{v}$ the *Von Neumann-Morgenstern utility function* inducing $\dot{u}$ ($\ddot{v}$ is obviously unique). Denote by $\mathcal{F}_{EU}(\Delta(\mathcal{Z}) \to \mathbb{R})$ the set of $\mathcal{Z}$-expected utility functions. Given $\dot{\succsim} \in \mathcal{B}(\Delta(\mathcal{Z}))$, denote by $\mathcal{R}_{EU}(\dot{\succsim})$ the set of *expected utility representations* of $\dot{\succsim}$, i.e. $\mathcal{Z}$-expected utility functions that are utility representations of $\dot{\succsim}$. Let

$$\mathcal{B}_{EU}(\Delta(\mathcal{Z})) = \{\dot{\succsim} \in \mathcal{B}(\Delta(\mathcal{Z})) : \mathcal{R}_{EU}(\dot{\succsim}) \neq \emptyset\}.$$

Clearly, given $\dot{\succsim} \in \mathcal{B}(\Delta(\mathcal{Z}))$, $\mathcal{R}_{EU}(\dot{\succsim}) \subseteq \mathcal{R}_U(\dot{\succsim})$, hence $\mathcal{B}_{EU}(\Delta(\mathcal{Z})) \subseteq \mathcal{B}_U(\Delta(\mathcal{Z}))$. Von Neumann and Morgenstern (1944) axiomatized $\mathcal{B}_{EU}(\Delta(\mathcal{Z}))$, and showed that an expected utility representation of $\dot{\succsim} \in \mathcal{B}(\Delta(\mathcal{Z}))$ is unique up to a positive affine transformation, i.e. given $\dot{u} \in \mathcal{R}_{EU}(\dot{\succsim})$ and $\dot{u}' \in \mathcal{F}(\Delta(\mathcal{Z}) \to \mathbb{R})$, $\dot{u}' \in \mathcal{R}_{EU}(\dot{\succsim})$ if and only if $\exists(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}$ such that $\dot{u}' = \alpha\dot{u} + \beta$.

Definition. *Assume $\mathcal{A}$ satisfies Structural Axiom (F^{DLR}). An* ***expected utility subjective state space representation*** *of $\succsim \in \mathcal{B}(\mathcal{A})$ is a subjective state space representation $(\dot{U}, g)$ of $\succsim$ such that $\dot{U} \in \mathcal{P}(\mathcal{F}_{EU}(\Delta(\mathcal{Z}) \to \mathbb{R}))$ and $\dot{U}$ is relevant.*

Given $\succsim \in \mathcal{B}(\mathcal{A})$, denote by $\mathcal{R}_{EUSSS}(\succsim)$ the set of expected utility subjective state space representations of $\succsim$. Dekel, Lipman, and Rustichini (2001, Theorems 1 and 3) showed that an expected utility subjective state space representation $(\dot{U}, g)$ of $\succsim \in \mathcal{B}(\mathcal{A})$ such that $u_{\dot{U},g}$ is continuous is essentially unique (in particular, $\dot{P}(\dot{U})$ is unique), and axiomatized the existence of such a representation.[9]

In order to relate the BC-preferences model to the literature on preference for flexibility, I shall from now on interpret the primitive concept $\succsim$ of this literature as being the agent's behavioral preference relation $\succsim_B$ on $\mathcal{A}$. This interpretation of $\succsim$ is arguably the natural one in this literature, and is at least plausible since the existence of a subjective state space representation implies completeness of $\succsim$. Since $\mathcal{B}_{SSS}(\mathcal{A}) \subset \mathcal{B}_{C^+,F^+}(\mathcal{A})$, both Kreps's (1979) and Dekel, Lipman, and Rustichini's (2001) models are less general than the BC-preferences model (in either of its two versions under strong consistency). Moreover, one can check that if a subjective state space representation $(\dot{U}, g)$ of $\succsim_B$ exists, then the cognitive preference relation $\succsim_C$ on $\mathcal{A}$ derived from $\succsim_B$ by way of Theorem 3 is such that $\forall a, a' \in \mathcal{A}$,

$$a \succsim_C a' \Leftrightarrow [\forall \dot{u} \in \dot{U},\ \sup_{x \in a} \dot{u}(x) \geq \sup_{x' \in a'} \dot{u}(x')], \tag{1}$$

which, if $\dot{u}(a)$ and $\dot{u}(a')$ are compact, is equivalent to

$$a \succsim_C a' \Leftrightarrow [\forall \dot{\succsim} \in \dot{P}(\dot{U}),\ \forall x' \in a',\ \exists x \in a \text{ such that } x \dot{\succsim} x']. \tag{2}$$

(2) precisely models the intuitive justification for Condition F-DP given in Section 3, provided that $\succsim_C$ is transitive. If $\succsim_C$ is not transitive, then this intuitive justification can be modelled similarly, only with $\dot{\succsim}$ depending on the pair $\{a, a'\}$ which is considered; one can check that this modelling is equivalent to Condition F-DP. Note that $\succsim_C$ is entirely determined by $\dot{U}$, independently of g; more precisely, (1) implies that $U \in \mathcal{R}_{MU}(\succsim_C)$, where $U = \{u_{\dot{u}} : \dot{u} \in \dot{U}\}$ and $\forall \dot{u} \in \dot{U}$, $u_{\dot{u}} \in \mathcal{F}(\mathcal{A} \to \mathbb{R})$ is defined by $\forall a, a' \in \mathcal{A}$, $u_{\dot{u}}(a) = \sup_{x \in a} \dot{u}(x)$.

The BC-preferences model shows that Kreps's (1979) axioms are not necessary for the existence of $\succsim_C$, and neither are Dekel, Lipman, and Rustichini's (2001) additional axioms necessary for its uniqueness. On the other hand, these axioms yield the "aggregator" g, which determines the agent's choice behavior concerning

[9]This representation concept, which they called "ordinal EU representation", is not the most general they axiomatized, as they also defined a "weak EU representation", in which g need not be increasing.

cognitively incomparable alternatives, as well as the subjective state space $\dot{P}(\dot{U})$, which seems to describe her tastes more precisely than her cognitive preference relation $\succsim_C$ does. To make this latter point precise, I shall now investigate the conditions under which two behavioral preference relations elicit the same cognitive preference relations, but different subjective state spaces. To this end, the following notation is needed. Given $\dot{P} \in \mathcal{P}(\mathcal{B}(\Delta(\mathcal{Z})))$, let

$$\mathcal{R}_{EU}(\dot{P}) = \{\dot{\mathbf{u}} \in \mathcal{F}(\dot{P} \to (\mathcal{F}(\Delta(\mathcal{Z}) \to \mathbb{R}))) : \forall \dot{\succsim} \in \dot{P},\ \dot{\mathbf{u}}(\dot{\succsim}) \in \mathcal{R}_{EU}(\dot{\succsim})\},$$

i.e. $\mathcal{R}_{EU}(\dot{P})$ is the set of functions mapping each $\dot{\succsim} \in \dot{P}$ into an expected utility representation of $\dot{\succsim}$. Given $\dot{U} \in \mathcal{P}(\mathcal{F}(\Delta(\mathcal{Z}) \to \mathbb{R}))$, call *convex hull* of $\dot{U}$ the set

$$co(\dot{U}) = \{\sum\nolimits_{\dot{u} \in \dot{U}} \mu(\dot{u})\dot{u} : \mu \in \Delta(\dot{U})\},$$

i.e. the set of convex combinations of elements of $\dot{U}$. Clearly, $\dot{U} \subseteq co(\dot{U})$. Given $\dot{P} \in \mathcal{P}(\mathcal{B}_{EU}(\Delta(\mathcal{Z})))$, call *convex hull* of $\dot{P}$ the set

$$co(\dot{P}) = \{\dot{\succsim} \in \mathcal{B}(\Delta(\mathcal{Z})) : \exists \dot{\mathbf{u}} \in \mathcal{R}_{EU}(\dot{P}) \text{ such that } co(\dot{\mathbf{u}}(\dot{P})) \cap \mathcal{R}_{EU}(\dot{\succsim}) \neq \emptyset\},$$

i.e. the set of binary relations on $\Delta(\mathcal{Z})$ that admit as an expected utility representation a convex combination of expected utility representations of the elements of $\dot{P}$. Clearly, $\dot{P} \subseteq co(\dot{P}) \subseteq \mathcal{P}(\mathcal{B}_{EU}(\Delta(\mathcal{Z})))$.

Theorem 4. *Assume $\mathcal{A}$ satisfies Structural Axiom (F^{DLR}). Let $\succsim_{B_1}$ and $\succsim_{B_2}$ be two behavioral preference relations on $\mathcal{A}$. $\forall j \in \{1,2\}$, let $(\dot{U}_j, g_j) \in \mathcal{R}_{EUSSS}(\succsim_{B_j})$ such that $\dot{P}(\dot{U}_j)$ is finite, and $\succsim_{C_j} \in \mathcal{R}_{C^+,F^+}(\succsim_{B_j})$. Then $\succsim_{C_1} = \succsim_{C_2}$ if and only if $co(\dot{P}(\dot{U}_1)) = co(\dot{P}(\dot{U}_2))$.*

Thus, two agents whose behavioral preference relations elicit two distinct expected utility subjective state spaces which have the same convex hull are indistinguishable, as far as welfare is concerned, since their behavioral preference relations elicit the same cognitive preference relation. Of course, the two agents have different choice behaviors, but they have the same cognitive preferences, and only adopt different behaviors when forced to choose between cognitively incomparable alternatives.

5 Weak consistency

This last section is devoted to the final stage of the BC-preferences model, i.e. to solving the uniqueness and existence problems under weak consistency. The results to be shown generalize those of Section 3, as well as results I proved in Danan (2002). Intuitively, one should now have in mind the purely behavioral interpretation of $\succsim_B$; however, as explained in Section 2, behavioral preferences will not be assumed to be antisymmetric. First some mathematical notation. Call *ordered* (resp. *partially ordered*) *space* any couple $(\mathcal{S}, \geq_{\mathcal{S}})$ such that $\mathcal{S}$ is a nonempty set and $\geq_{\mathcal{S}}$ is a complete (resp. reflexive), antisymmetric, and transitive binary relation on $\mathcal{S}$. Say that a partially ordered space $(\mathcal{S}, \geq_{\mathcal{S}})$ is

— *unbounded* if $\forall \tilde{s} \in \mathcal{S}$, $\exists s, s' \in \mathcal{S}$ such that $s <_{\mathcal{S}} \tilde{s} <_{\mathcal{S}} s'$,

— *dense* if $\forall s, s' \in \mathcal{S}$ such that $s <_{\mathcal{S}} s'$, $\exists \tilde{s} \in \mathcal{S}$ such that $s <_{\mathcal{S}} \tilde{s} <_{\mathcal{S}} s'$.

Note that if $(\mathcal{S}, \geq_{\mathcal{S}})$ is unbounded, then $\mathcal{S}$ is infinite. Clearly, $(\mathbb{R}, \geq)$ is an unbounded and dense ordered space.

Does Condition F-LA (resp. F-DP) still solve the uniqueness problem under weak consistency? Given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, denote by $\mathcal{R}_{C^-,F^-}(\succsim_B)$ (resp. $\mathcal{R}_{C^-,F^+}(\succsim_B)$) the set of $\succsim_C \in \mathcal{R}_{C^-}(\succsim_B)$ satisfying Condition F-LA (resp. F-DP). By Lemma 2b, $\mathcal{R}_{C^-,F^+}(\succsim_B) \subseteq \mathcal{R}_{C^-,F^-}(\succsim_B)$. The answer to the latter question is negative, as one can check that $\mathcal{R}_{C^+,F^+}(\succsim_B) \cup \{\Theta_{\mathcal{A}}\} \subseteq \mathcal{R}_{C^-,F^+}(\succsim_B)$. Intuitively, when strong consistency is replaced by weak consistency, it becomes conceivable that all alternatives are cognitively indifferent. Hence, because of the unobservability of cognitive indifference, additional conditions are needed under weak consistency in order to solve the uniqueness problem.

Since the flexibility structural framework seems to be of no help for this problem, I shall impose an additional structural axiom, under which the desired conditions can be formally stated and intuitively justified. It turns out to be sufficient to formalize Savage's (1954) suggestion of adding monetary bonuses to the alternatives (see Section 2). Formally, the following structural axiom is imposed.

Structural Axiom (M: Money). *There exists a couple $(\Phi, (\geq_\phi)_{\phi\in\Phi})$ such that Φ is a partition of $\mathcal{A}$ and $\forall \phi \in \Phi$, $(\phi, \geq_\phi)$ is an unbounded and dense ordered space.*

Each $\phi \in \Phi$ is interpreted as a set of alternatives which only differ by their attached monetary bonus. Which of two alternatives in ϕ features the highest bonus is determined by the order $\geq_\phi$. Structural Axiom M is essentially unrestrictive, since given any nonempty set $\dot{\mathcal{A}}$, its extension $\dot{\mathcal{A}} \times \mathbb{R}$ obviously satisfies it. This extension merely consists in explicitly attaching monetary bonuses to the alternatives in $\dot{\mathcal{A}}$. In order to get a smaller extension, one can replace $\mathbb{R}$ by any open interval of $\mathbb{R}$. Given $a \in \mathcal{A}$, denote by $\phi(a)$ the element of Φ containing a. Define the binary relation $\geq_{\mathcal{A}}$ on $\mathcal{A}$ by $\forall a, a' \in \mathcal{A}$,

$$a \geq_{\mathcal{A}} a' \Leftrightarrow [\phi(a) = \phi(a') \text{ and } a \geq_{\phi(a)} a'].$$

$a \geq_{\mathcal{A}} a'$ is interpreted as "*a weakly M-dominates* a'". One can check that $(\mathcal{A}, \geq_{\mathcal{A}})$ is an unbounded and dense partially ordered space. Unboundedness of $(\mathcal{A}, \geq_{\mathcal{A}})$ means that a (positive or negative) monetary bonus can be added to any alternative, and denseness of $(\mathcal{A}, \geq_{\mathcal{A}})$ that this monetary bonus can be taken arbitrarily small.

Given $\curvearrowright \in \mathcal{B}(\mathcal{A})$, define $\uparrow\curvearrowright, \downarrow\curvearrowright, \rceil\curvearrowright, \rfloor\curvearrowright \in \mathcal{B}(\mathcal{A})$ by $\forall a, a' \in \mathcal{A}$,

$$\begin{aligned} a \uparrow\curvearrowright a' &\Leftrightarrow [\forall \tilde{a} >_{\mathcal{A}} a,\ \tilde{a} \curvearrowright a'], \qquad & a \rceil\curvearrowright a' &\Leftrightarrow [\exists \tilde{a} >_{\mathcal{A}} a \text{ such that } \tilde{a} \curvearrowright a'], \\ a \downarrow\curvearrowright a' &\Leftrightarrow [\forall \tilde{a} <_{\mathcal{A}} a,\ \tilde{a} \curvearrowright a'], \qquad & a \rfloor\curvearrowright a' &\Leftrightarrow [\exists \tilde{a} <_{\mathcal{A}} a \text{ such that } \tilde{a} \curvearrowright a']. \end{aligned}$$

Visualizing this notation might help following the remainder of this section: each $\phi \in \Phi$ can be represented by a vertical axis which increases with the monetary bonus. Then, given $a, a' \in \mathcal{A}$ (which are not necessarily on the same axis), $a \uparrow\curvearrowright a'$ means that any alternative strictly above a (on a's axis) bears the relation $\curvearrowright$ to a', while $a \rceil\curvearrowright a'$ means that there exists an alternative strictly above a which bears the relation $\curvearrowright$ to a'. $a \downarrow\curvearrowright a'$ and $a \rfloor\curvearrowright a'$ can be visualized similarly. Figure 2 illustrates the following example: $\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, $a_1 \curvearrowright a_2$, $a'_2 \uparrow\curvearrowright a_3$, and $a_4 \rceil\curvearrowright a'_3$. Complement and dual notation, such as $a \uparrow\not\curvearrowright a'$, $a \curvearrowleft\lfloor a'$, or $a \curvearrowright\downarrow a'$, will also be used.

In Danan (2002, Theorem 1), I showed that the three following conditions solve the uniqueness problem under weak consistency and the additional condition that cognitive preferences are complete.

Condition (M-M: Monotonicity). $\succsim_C \subseteq \uparrow\succ_C$.

Condition (M-SC: Strong Continuity). $\uparrow\succ_C \subseteq \succsim_C$.

Condition (M-S: Symmetry). $\uparrow\succ_C = \succ_C\downarrow$.

These conditions can be intuitively justified by an interpretive assumption drawing an analogy between the agent's cognitive preferences and a subjective "balance"

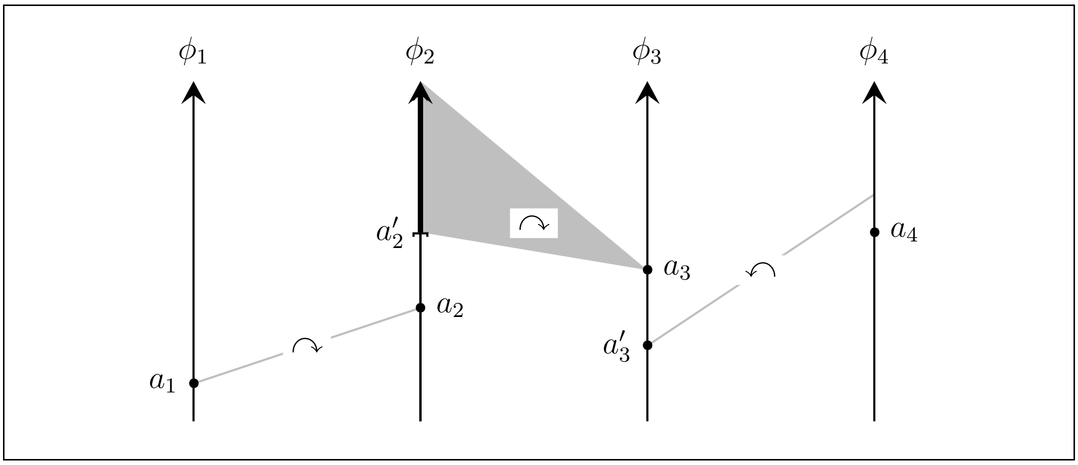

Figure 2: Graphical representation of a monetary structure

(call it the *balance* assumption): the agent weakly prefers a to a' when a weighs more than a' on her subjective balance. Monetary bonuses can then be thought of as weights that can be added to an alternative or subtracted from it on the balance, the assumption being that the agent always desires more money. It is important to understand that although a balance can be assumed to induce a complete and transitive ranking of all alternatives, the balance assumption implies neither completeness nor transitivity of the agent's cognitive preferences. This is because cognitive preferences are a matter of introspection, i.e. of the agent trying to determine which of two alternatives would weigh the most if they were compared by means of her subjective balance. Hence, first, she may be unable to determine which alternative weighs the most (or put differently, to perfectly observe her subjective balance), in which case her cognitive preferences are incomplete, and second, she may use different balances depending on which pair $\{a, a'\}$ she is considering, in which case her cognitive preferences may be intransitive. However, according to the balance assumption, a specificity of money (besides being always desired) is that adding money to an alternative amounts to adding weight on a given balance, and does not induce the agent to use a different balance.

Condition M-M asserts that if a is cognitively weakly preferred to a', then any alternative strictly M-dominating a is cognitively strictly preferred to a'. Intuitively, if a weighs at least as much as a', then any weight added to a makes it weigh strictly more than a'. Condition M-SC asserts that if any alternative strictly M-dominating a is cognitively strictly preferred to a', then a is cognitively weakly preferred to a'. Intuitively, if any weight added to a makes it weigh strictly more than a', then, first, a' does not weigh strictly more than a, for otherwise a small weight added to a would leave it weighing strictly less than a' (recall that weights can be taken arbitrarily small), and second, the agent knows which alternative weighs the most, for otherwise she would consider possible that a' weighs strictly more than a, and hence that a small weight added to a leaves it weighing strictly less than a'. Condition M-S asserts that any alternative strictly M-dominating a is cognitively strictly preferred to a' if and only if a is cognitively strictly preferred to any alternative strictly M-dominated by a'. Intuitively, if any weight added to a makes it weigh strictly more than a', then a weighs at least as much as a' by Condition M-SC, hence any weight subtracted from a' makes it weigh strictly less than a. Note that since $(\mathcal{A}, \geq_{\mathcal{A}})$ is dense, it follows from Condition M-M that $\uparrow\succsim_C = \uparrow\succ_C$, and hence from Condition M-S that $\succsim_C\downarrow = \succ_C\downarrow$. Note also that Conditions M-M and M-S would not make sense if $(\mathcal{A}, \geq_{\mathcal{A}})$ were not unbounded, for they would then force any upper bound of some $\phi \in \Phi$ to be cognitively weakly preferred to any other alternative, and strictly preferred to any alternative which is not an upper bound.

The reason why Condition M-SC is called a continuity condition is that it can be topologically restated as such. To this end, endow each $\phi \in \Phi$ with the $\geq_\phi$-order topology, and $\forall \phi, \phi' \in \Phi$, endow each $\phi \times \phi'$ with the product topology. In Danan (2002, Lemma 2), I showed that if a binary relation $\succsim_C$ on $\mathcal{A}$ satisfies Conditions M-M and M-S, then it satisfies Condition M-SC if and only if $\forall \phi, \phi' \in \Phi$, $\succsim_C \cap (\phi \times \phi')$ is closed in $\phi \times \phi'$. Another usual continuity condition is that $\succ_C \cap (\phi \times \phi')$ is open in $\phi \times \phi'$. These two continuity properties are clearly equivalent if $\succsim_C$ is complete, but they are independent if $\succsim_C$ is incomplete. Furthermore, since Conditions M-M and M-S imply that $\succsim_C|_{\phi \times \phi'}$ is transitive (Danan 2002, Lemma 1), it follows from Schmeidler (1971) that their conjunction forces completeness of $\succsim_C$. Hence a choice has to be made between the two versions of continuity; the "closed" version is chosen here since in the literature on incomplete preferences, it is the one naturally associated with continuous representations: one can check that if $\succsim_C$ admits a multi-utility representation U such that each $u \in U$ is continuous, then it satisfies the closed version of continuity. Hence if $\succsim_C$ is incomplete and satisfies Axioms M-M, M-SC, and M-S, then $\exists a, a' \in \mathcal{A}$ such that $a \succ_C a'$, but any small positive bonus added to a' makes the two alternatives cognitively incomparable. In this case, cognitive strict preference should not be interpreted in the usual way, but rather as "the agent desires a at least as much as a', but does not know whether she desires a strictly more than a' or desires the two alternatives equally".

Combining these monetary conditions with Condition F-LA or F-DP presupposes that Structural Axioms F and M are simultaneously imposed, hence it is of interest to investigate whether the conjunction of these two structural axioms is essentially unrestrictive. Given an arbitrary set $\dot{\mathcal{A}}$, the straightforward extensions which are candidates for satisfying Structural Axioms F and M are $\mathcal{P}(\dot{\mathcal{A}}) \times \mathbb{R}$ and $\mathcal{P}(\dot{\mathcal{A}} \times \mathbb{R})$. However, one can check that the former violates Structural Axiom F and the latter Structural Axiom M. Yet the conjunction of Structural Axioms F' (hence F) and M is essentially unrestrictive, as the following lemma shows.

Lemma 4. *Given a nonempty set $\dot{\mathcal{A}}$, let*

$$\mathcal{A}_1 = \{a \in \mathcal{P}(\dot{\mathcal{A}} \times \mathbb{R}) : \forall \dot{a} \in \dot{\mathcal{A}},\ \{m \in \mathbb{R} : (\dot{a}, m) \in a\} \textit{ is bounded}\}, \tag{3}$$

$$\mathcal{A}_2 = \{a \in \mathcal{P}(\dot{\mathcal{A}} \times \mathbb{R}) : \forall \dot{a} \in \dot{\mathcal{A}},\ \#\{m \in \mathbb{R} : (\dot{a}, m) \in a\} \leq 1\}. \tag{4}$$

Then

a. $\mathcal{A}_1$ satisfies Structural Axioms F' and M,

b. $\mathcal{A}_2$ satisfies Structural Axioms F and M.

In order to use smaller extensions, one can replace boundedness by compactness or finiteness in (3). But $\mathcal{A}_2$ is an even smaller extension (note that it violates Structural Axiom F' but satisfies F, which is the motivation for having introduced this latter structural axiom in Section 3). Finally, it is possible to replace $\mathbb{R}$ by any open interval of $\mathbb{R}$; in this case, one must add to (3) (resp. (4)) the requirement that $\{m \in \mathbb{R} : \exists \dot{a} \in \dot{\mathcal{A}}$ such that $(\dot{a}, m) \in a\}$ is bounded.

Unfortunately, Conditions F-LA (resp. F-DP), M-M, M-SC, and M-S do not solve the uniqueness problem under weak consistency. For example, let $\dot{\mathcal{A}}$ be a nonempty, finite set, and consider the set of alternatives $\mathcal{A}_2$ defined by (4), endowed with the operator $\cup_{\mathcal{A}_2}$ defined in the proof of Lemma 4b (thus $\mathcal{A}_2$ satisfies Structural Axioms F and M), i.e. $\forall a, a' \in \mathcal{A}_2$,

$$a \cup_{\mathcal{A}_2} a' = \{(\dot{a}, m) \in a \cup a' : \forall \varepsilon \in \mathbb{R}_+^*,\ (a, m + \varepsilon) \notin a \cup a'\}.$$

Define $\rho, \tau \in \mathcal{F}(\mathcal{A}_2 \to \mathbb{R})$ by $\forall a \in \mathcal{A}_2$,

$$\rho(a) = \max\nolimits_{(\dot{a},m)\in a} m, \qquad \tau(a) = \sum\nolimits_{(\dot{a},m)\in a} \exp(m),$$

and $\succsim_B, \succsim_{C_1}, \succsim_{C_2} \in \mathcal{B}(\mathcal{A}_2)$ by $\forall a, a' \in \mathcal{A}_2$,

$$\begin{aligned} a \succsim_B a' &\Leftrightarrow [\rho(a) > \rho(a') \text{ or } [\rho(a) = \rho(a') \text{ and } \tau(a) \geq \tau(a')]], \\ a \succsim_{C_1} a' &\Leftrightarrow \rho(a) \geq \rho(a'), \\ a \succsim_{C_2} a' &\Leftrightarrow [\forall (\dot{a}', m') \in a',\ \exists \varepsilon \in \mathbb{R}_+ \text{ such that } (\dot{a}', m' + \varepsilon) \in a]. \end{aligned}$$

Then one can check that $\forall j \in \{1, 2\}$, $(\succsim_B, \succsim_{C_j})$ is a weakly consistent BC-preference system and $\succsim_{C_j}$ satisfies Conditions F-DP, M-M, M-SC, and M-S. These two cognitive preference relations are sensible *per se*: $\succsim_{C_1}$ ranks all opportunity sets according to their maximal monetary bonus (hence is complete), while $\succsim_{C_2}$ only ranks two opportunity sets if one of them consists in options which are weakly M-dominated by some option in the other (hence is very incomplete). Hence I shall not look for additional conditions, but rather impose the following linking condition.

Definition. *Assume $\mathcal{A}$ satisfies Structural Axiom M. A BC-preference system $(\succsim_B, \succsim_C)$ on $\mathcal{A}$ is **M-regular** if $\succ_C \subseteq \succsim_B \lceil$.*

M-regularity asserts that if a is cognitively strictly preferred to a', then there exists an alternative strictly M-dominating a' to which a is behaviorally weakly preferred. One can check that in the latter example, $(\succsim_B, \succsim_{C_1})$ is M-regular, but $(\succsim_B, \succsim_{C_2})$ is not: for example, given $\dot{a}, \dot{a}' \in \dot{\mathcal{A}}$ such that $\dot{a} \neq \dot{a}'$, one has $\{(\dot{a}, 0), (\dot{a}', 0)\} \succ_C \{(\dot{a}, 0)\}$ since the former opportunity set is strictly more flexible than the latter, but $\forall m \in \mathbb{R}_+^*$, $\{(\dot{a}, m)\} \succ_B \{(\dot{a}, 0), (\dot{a}', 0)\}$. Intuitively, the agent is unable to cognitively trade off money for flexibility, so $\forall m \in \mathbb{R}_+^*$, $\{(\dot{a}, m)\} \bowtie_{C_2} \{(\dot{a}, 0), (\dot{a}', 0)\}$; this, *per se* does not violate M-regularity, but the fact that she never behaviorally trades off money for flexibility does. The interpretive assumption justifying M-regularity is that cognitive strict preference has some "substance", in the sense that if the agent cognitively strictly prefers a to a', then not only does she choose a over a' (as required by weak consistency), but also a small positive bonus added to a' does not reverse this behavior, even if it makes the two alternatives cognitively incomparable (call it the *substance* assumption).

As under strong consistency, Conditions F-LA and F-DP can be restated as linking conditions. Under weak consistency, this requires the following notation. Given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, define $\|_B^*, \Vdash_B^* \in \mathcal{B}(\mathcal{A})$ by $\forall a, a' \in$

$\mathcal{A}$,

$$a \parallel^*_B a' \Leftrightarrow a \rceil\precsim_B a \cup_{\mathcal{A}} a' \succsim_B \lceil a', \qquad a \Vdash^*_B a' \Leftrightarrow a \uparrow\succ_B a \cup_{\mathcal{A}} a' \uparrow\succ_B a.$$

$a \parallel^*_B a'$ is interpreted as "the agent has a behavioral *M-preference for flexibility* at $\{a, a'\}$", and $a \Vdash^*_B a'$ as "the agent has a behavioral *M-indifference to flexibility* at (a, a')". $\parallel^*_B$ and $\Vdash^*_B$ are the natural analogs of $\parallel_B$ and $\Vdash_B$ under weak consistency: since cognitive indifference is unobservable, one may observe $a \parallel_B a'$ while $a \sim_C a \cup_{\mathcal{A}} a' \sim_C a'$, hence cognitive strict preference is rather identified with the acceptance to give up some positive monetary bonus; similarly, cognitive weak preference is identified with the acceptance of an arbitrarily small positive monetary bonus. Note that this intuition is only valid since one alternative is more flexible than the other, as in this case the two alternatives are cognitively comparable by Condition F-LA. The following lemma is the analog of Lemma 3 under weak consistency.

Lemma 5. *Assume $\mathcal{A}$ satisfies Structural Axioms F and M. Let $(\succsim_B, \succsim_C)$ be a weakly consistent and M-regular BC-preference system on $\mathcal{A}$ such that $\succsim_C$ satisfies Condition M-M. Then*

- *a. if $\succsim_C$ is F-complete, then $\forall a, a' \in \mathcal{A}$, $a \succsim_C a \cup_{\mathcal{A}} a' \Leftrightarrow a \uparrow\succ_B a \cup_{\mathcal{A}} a'$ and $a \cup_{\mathcal{A}} a' \succsim_C a \Leftrightarrow a \cup_{\mathcal{A}} a' \uparrow\succ_B a$,*
- *b. $\succsim_C$ satisfies Condition F-LA if and only if [$\bowtie_C = \parallel^*_B$ and $\parallel^*_B$ is F-empty],*
- *c. $\succsim_C$ satisfies Condition F-DP if and only if [$\succsim_C = \Vdash^*_B$ and $\Vdash^*_B$ is F-complete].*

The following theorem shows that M-regularity and Conditions F-LA and M-M solve the uniqueness problem under weak consistency.

Theorem 5. *Assume $\mathcal{A}$ satisfies Structural Axioms F and M. Let $\succsim_B$ be a behavioral preference relation on $\mathcal{A}$, and $\succsim_{C_1}$ and $\succsim_{C_2}$ be two cognitive preference relations on $\mathcal{A}$ satisfying Conditions F-LA and M-M such that $\forall j \in \{1, 2\}$, $(\succsim_B, \succsim_{C_j})$ is a strongly consistent and M-regular BC-preference system. Then $\succsim_{C_1} = \succsim_{C_2}$.*

It may be surprising that Condition M-SC is not necessary for solving the uniqueness problem, as this condition is crucial when $\succsim_C$ is complete (Danan 2002, Theorem 1). However, M-regularity implies the following continuity condition, which is weaker than Condition M-SC in general, and equivalent if $\succsim_C$ is complete, as the next lemma shows.

Condition (M-WC: Weak Continuity). $\prec_C \cap \uparrow\succ_C = \emptyset$.

Lemma 6. *Assume $\mathcal{A}$ satisfies Structural Axiom M.*

a. If a binary relation $\succsim_C$ on $\mathcal{A}$ satisfies Condition M-SC, then it satisfies Condition M-WC; moreover, the converse holds if $\succsim_C$ is complete,

b. If a weakly consistent BC-preference system $(\succsim_B, \succsim_C)$ on $\mathcal{A}$ is M-regular, then $\succsim_C$ satisfies Condition M-WC; moreover, the converse holds if $\succsim_C$ is complete and satisfies Condition M-M.

Thus, if $\succsim_C$ is complete, then it is equivalent to impose Condition M-SC, or M-WC, or M-regularity, but if $\succsim_C$ is incomplete, then M-regularity turns out to be the crucial property. Condition M-WC asserts that if the agent cognitively strictly prefers a' to a, then an arbitrarily small positive bonus added to a does not make it cognitively strictly preferred to a'. It is justified by the balance assumption since it is weaker than Condition M-SC. Like Condition M-SC, it can be topologically restated. To this end, say that two subsets of a topological space are *separated* if none intersects the other's closure.

Lemma 7. *Assume $\mathcal{A}$ satisfies Structural Axiom M. Let $\succsim_C$ be a binary relation on $\mathcal{A}$ satisfying Conditions M-M and M-S. Then $\succsim_C$ satisfies Condition M-WC if and only if $\forall \phi, \phi' \in \Phi$, $\succ_C \cap(\phi \times \phi')$ and $\prec_C \cap(\phi \times \phi')$ are separated in $\phi \times \phi'$.*

As under strong consistency, one can consider two versions of the BC-preferences model under weak consistency, depending on whether one imposes Condition F-LA or F-DP. Furthermore, one has the possibility of imposing Condition M-SC, which is formally unnecessary for solving the uniqueness problem, but intuitively justified by the balance assumption. These two modelling choices give rise to four possible versions of the BC-preferences model under weak consistency. I shall, however, restrict attention to the two "extreme" ones, i.e. the most general and the most intuitive. Formally, given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, denote by $\mathcal{R}_{C^-,F^-,M^-}(\succsim_B)$ (resp. $\mathcal{R}_{C^-,F^+,M^+}(\succsim_B)$) the set of $\succsim_C \in \mathcal{R}_{C^-}(\succsim_B)$ satisfying Conditions F-LA, M-M, and M-S (resp. F-DP, M-M, M-SC, and M-S), such that $(\succsim_B, \succsim_C)$ is M-regular. Let

$$\mathcal{B}_{C^-,F^-,M^-}(\mathcal{A}) = \{\succsim_B \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{C^-,F^-,M^-}(\succsim_B) \neq \emptyset\},$$
$$\mathcal{B}_{C^-,F^+,M^+}(\mathcal{A}) = \{\succsim_B \in \mathcal{B}(\mathcal{A}) : \mathcal{R}_{C^-,F^+,M^+}(\succsim_B) \neq \emptyset\},$$

Obviously, $\mathcal{B}_{C^-,F^+,M^+}(\mathcal{A}) \subseteq \mathcal{B}_{C^-,F^-,M^-}(\mathcal{A})$. In order to state the axioms to be imposed, the following notation will be convenient. Given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, define $\succsim_B^*, \succsim_B^\circ \in \mathcal{B}(\mathcal{A})$ by

$$\succsim_B^* = \succsim_B \setminus \|_B^*, \qquad\qquad \succsim_B^\circ = \uparrow\succ_B .$$

$a \succsim_B^* a'$ means that the agent chooses a over a' and does not have a behavioral M-preference for flexibility at $\{a, a'\}$. One can check that $\succ_B^* = \succ_B \setminus \|_B^*$, $\sim_B^* = \sim_B \setminus \|_B^*$, and $\bowtie_B^* = \|_B^*$. Say that $\succsim_B$ is *FM-complete** if $\succsim_B^*$ is F-complete (i.e. $\|_B^*$ is F-empty). The three following axioms turn out to solve the existence problem in the general version of the BC-preferences model under weak consistency.

Axiom (FM-M*: Monotonicity). $\succsim_B^* \subseteq \uparrow\succ_B^*$.

Axiom (FM-S*: Symmetry). $\uparrow\succ_B^* = \succ_B^*\downarrow$.

Axiom (FM-R*: Regularity). $\not\|_B^* \cap (\|_B^* \lceil) \subseteq \succsim_B \lceil$.

Axioms FM-M* and FM-S* are respectively similar to Conditions M-M and M-S, only they involve $\succsim_B^*$ instead of $\succsim_C$. Axiom FM-R* asserts that if the agent does not have a behavioral M-preference for flexibility at $\{a, a'\}$ but has a behavioral M-preference for flexibility at $\{a, \tilde{a}'\}$, for some $\tilde{a}'$ strictly M-dominating a', then there exists an alternative strictly M-dominating a' to which a is behaviorally weakly preferred. Its name is justified by the fact that if $\bowtie_C = \|_B^*$ and $\succsim_C$ satisfies Condition M-M, then $\not\|_B^* \cap (\|_B^* \lceil) \subseteq \succ_C$, hence Axiom FM-R* is implied by M-regularity.

Theorem 6. *Assume $\mathcal{A}$ satisfies Structural Axioms F and M. Let $\succsim_B$ be a behavioral preference relation on $\mathcal{A}$. Then $\succsim_B$ is FM-complete* and satisfies Axioms FM-M*, FM-S*, and FM-R* if and only if there exists a cognitive preference relation $\succsim_C$ on $\mathcal{A}$ satisfying Conditions F-LA, M-M, and M-S such that $(\succsim_B, \succsim_C)$ is a weakly consistent and M-regular BC-preference system.*

In the intuitive version of the model, the existence problem turns out to be solved by the two following additional axioms.

Axiom (FM-SC*: Strong Continuity). $\uparrow\not\|_B^* \subseteq \not\|_B^*$.

Axiom (FM-M$^\circ$: Monotonicity). $\forall a, a' \in \mathcal{A}$, $a \cup_{\mathcal{A}} a' \succsim_B^\circ a$.

Axiom (FM-T°: Transitivity). *$\forall a, a' \in \mathcal{A}$, $\succsim_B^\circ|_{\{a,a',a\cup_{\mathcal{A}}a'\}}$ is transitive.*

Axioms FM-M° and FM-T° are respectively similar to Conditions F-M and F-T, only they involve $\succsim_B^\circ$ instead of $\succsim_C$. Clearly, Axiom FM-M° implies FM-completeness* of $\succsim_B$. Axiom FM-SC* asserts that if for all $\tilde{a}$ strictly M-dominating a, the agent does not have a behavioral M-preference for flexibility at $\{\tilde{a}, a'\}$, then she does not have a behavioral M-preference for flexibility at $\{a, a'\}$. It can be topologically restated as follows.

Lemma 8. *Assume $\mathcal{A}$ satisfies Structural Axioms F and M. Let $\succsim_B$ be a binary relation on $\mathcal{A}$ satisfying Axioms FM-M* and FM-S*. Then $\succsim_B$ satisfies Axiom FM-SC* if and only if $\forall \phi, \phi' \in \Phi$, $\|_B^* \cap(\phi \times \phi')$ is open in $\phi \times \phi'$.*

Theorem 7. *Assume $\mathcal{A}$ satisfies Structural Axioms F and M. Let $\succsim_B$ be a behavioral preference relation on $\mathcal{A}$. Then $\succsim_B$ satisfies Axioms FM-M*, FM-SC*, FM-S*, FM-R*, FM-M°, and FM-T° if and only if there exists a cognitive preference relation $\succsim_C$ on $\mathcal{A}$ satisfying Conditions F-DP, M-M, M-SC, and M-S such that $(\succsim_B, \succsim_C)$ is a weakly consistent and M-regular BC-preference system.*

This section's results can be summarized as follows. Concerning the uniqueness problem, given a behavioral preference relation $\succsim_B$ on $\mathcal{A}$, $\#\mathcal{R}_{C^-,F^-,M^-}(\succsim_B) \leq 1$ by Theorem 5, hence $\#\mathcal{R}_{C^-,F^+,M^+}(\succsim_B) \leq 1$ by Lemma 2b. As for the existence problem, by Theorem 6, $\mathcal{B}_{C^-,F^-,M^-}(\mathcal{A})$ is the set of FM-complete* behavioral preferences on $\mathcal{A}$ satisfying Axioms FM-M*, FM-S*, and FM-R*, and by Theorem 7, $B_{C^-,F^+,M^+}(\mathcal{A})$ is the set of behavioral preference relations on $\mathcal{A}$ satisfying Axioms FM-M*, FM-SC*, FM-S*, FM-R*, FM-M°, and FM-T°. As under strong consistency, any additional condition imposed on $\succsim_C$ can easily be characterized in terms of a corresponding axiom imposed on $\succsim_B$, since $\succsim_C = \Vdash_B^*$ by Lemma 5c. Note that none of this section's results makes use of the assumptions that $(\mathcal{A}, \geq_{\mathcal{A}})$ is unbounded. In this sense, this structural axiom has the same status than the axiom that $\succsim_B$ is antisymmetric: it is necessary for the concepts and conditions that are introduced to be intuitively reasonable, but not for proving the formal results (however, note that Lemma 4 would be less general if unboundedness of $(\mathcal{A}, \geq_{\mathcal{A}})$ were not included in Structural Axiom M).

Finally, how does the BC-preferences model, under weak consistency, relate to the literatures mentioned in Section 4? Concerning Arlegi and Nieto's (2001) model, the answer is obvious: since they derived no concept of cognitive indifference, weak and strong consistency are equivalent in their framework. Concerning Eliaz and Ok's (2003) and Dekel, Lipman, and Rustichini's (2001) models, a closer look is needed. A first point is that both these models impose structural axioms that are neither weaker nor stronger than Structural Axioms F and M. If all these structural axioms hold, these two models are neither more nor less general than the BC-preferences model, since both versions of this latter model now impose axioms which make use of the monetary structural setting. Finally, if all these axioms are simultaneously satisfied, then weak and strong consistency are equivalent. Indeed, both Eliaz and Ok (2003) and Dekel, Lipman, and Rustichini (2001) imposed continuity axioms which imply $\uparrow\succ^*_B \subseteq \succsim^*_B$. Hence one has $\sim_C \subseteq (\uparrow\succ_C) \cap (\prec_C\uparrow) \subseteq (\uparrow\succ^*_B) \cap (\prec^*_B\uparrow) \subseteq \sim^*_B \subseteq \sim_B$. Thus, one is brought back to the BC-preferences model under strong consistency and its connections with existing literatures presented in Section 4.

6 Conclusion

In this paper, I have proposed the BC-preferences model, which derives an agent's cognitive preferences from her behavioral preferences. An interest of this model is that it behaviorally defines a concept of welfare which is robust to the potential incompleteness of tastes and unobservability of cognitive indifference. In particular, it provides the literature on incomplete preferences with behavioral foundations. There are at least three questions to put on the research agenda.

— if instead of a behavioral preference relation, one takes as primitive the more general concept of a *choice function* (not necessarily rationalizable by such a relation), which axioms characterize the existence of a cognitive preference relation satisfying the imposed conditions?

— what is the (experimental) *descriptive relevance* of the distinction between behavioral and cognitive preferences, and in particular of tastes' incompleteness?

— which *normative conditions* characterize a rational cognitive preference relation, i.e. one which is consistent with some rational behavioral preference relation (in the sense of the money pump criterion)?

These three questions are topics of ongoing research.

Appendix: proofs

Proof of Lemma 1. If $\succsim_C = \succsim_B \setminus \bowtie_C$, then $\succ_C = \succ_B \setminus \bowtie_C$ and $\sim_C = \sim_B \setminus \bowtie_C$, hence $(\succsim_B, \succsim_C)$ is strongly consistent. Conversely, if $(\succsim_B, \succsim_C)$ is strongly consistent, then $\succsim_C \subseteq \succsim_B \setminus \bowtie_C$. Suppose $\exists a, a' \in \mathcal{A}$ such that $a \succsim_B a'$ and $a \not\bowtie_C a'$, but $a \not\succsim_C a'$. Then $a' \succ_C a$, hence $a' \succ_B a$, a contradiction. $\square$

Proof of Lemma 2. Let $a, a' \in \mathcal{A}$.

a. By reflexivity, $a \cup_{\mathcal{A}} a' \sim_C a \cup_{\mathcal{A}} a' = (a \cup_{\mathcal{A}} a') \cup_{\mathcal{A}} a$, i.e. $a \cup_{\mathcal{A}} a' \succsim_C a$ by Condition F-DP.

b. It follows from a that $\succsim_C$ is F-complete, hence $a \,\|_C\, a' \Leftrightarrow a \not\succsim_C a \cup_{\mathcal{A}} a' \not\precsim_C a'$. By a again, this is equivalent to $a \not\sim_C a \cup_{\mathcal{A}} a'_C \not\sim a'$, i.e. $a \not\succsim_C a' \not\succsim_C a$ by Condition F-DP, i.e. $a \bowtie_C a'$. $\square$

Proof of Lemma 3. *a.* Let $a, a' \in \mathcal{A}$. By Lemma 1, $a \succsim_C a \cup_{\mathcal{A}} a' \Leftrightarrow [a' \succsim_B a \cup_{\mathcal{A}} a'$ and $a \not\bowtie_C a \cup_{\mathcal{A}} a']$. Since $\succsim_C$ is F-complete, this is equivalent to $a \succsim_B a \cup_{\mathcal{A}} a'$. Similarly, $a \cup_{\mathcal{A}} a' \succsim_C a \Leftrightarrow [a \cup_{\mathcal{A}} a' \succsim_C a$ and $a \cup_{\mathcal{A}} a' \not\bowtie_C a] \Leftrightarrow a \cup_{\mathcal{A}} a' \succsim_C a$.

b. Whether $\bowtie_C = \|_C$ or $\bowtie_C = \|_B$, $\succsim_C$ is F-complete, hence $\|_C = \|_B$ by a.

c. Whether $\succsim_C = \Vdash_C$ or $\succsim_C = \Vdash_B$, $\succsim_C$ is F-complete, hence $\Vdash_C = \Vdash_B$ by a. $\square$

Proof of Theorem 1. By Lemma 1, $\forall j \in \{1, 2\}$, $\succsim_{C_j} = \succsim_B \setminus \bowtie_{C_j}$. Furthermore, by Lemma 3b, $\forall j \in \{1, 2\}$, $\bowtie_{C_j} = \|_B$. Hence $\succsim_{C_1} = \succsim_{C_2}$. $\square$

Proof of Theorem 2. Let $\succsim_C = \succsim_B \setminus \|_B$. Then $\succsim_C$ is a cognitive preference relation since $\succsim_B$ is reflexive and $\|_B$ is irreflexive. Moreover, $\bowtie_C = \|_B$, hence $\succsim_C = \succsim_B \setminus \bowtie_C$, i.e. $(\succsim_B, \succsim_C)$ is a strongly consistent BC-preference system by Lemma 1. Hence $\succsim_C$ satisfies Condition F-LA by Lemma 3b. $\square$

Proof of Theorem 3. *Necessity.* Let $\succsim_C$ be a cognitive preference relation on $\mathcal{A}$ satisfying Condition F-DP such that $(\succsim_B, \succsim_C)$ is a strongly consistent BC-preference system. Then $\succsim_B$ satisfies Axiom F-M by Lemmas 2a and 3a. To prove that $\succsim_B$ satisfies Axiom F-T, let $a, a' \in \mathcal{A}$. If $a \cup_{\mathcal{A}} a' \succsim_B a \succsim_B a'$, then $a \cup_{\mathcal{A}} a' \succsim_B a'$ by Axiom F-M. If $a \succsim_B a \cup_{\mathcal{A}} a' \succsim_B a'$, then $a \sim_B a \cup_{\mathcal{A}} a'$ by Axiom F-M, i.e. $a \succsim_C a'$ by Lemma 3c, hence $a \succsim_B a'$ by strong consistency. Finally, if $a \succsim_B a' \succsim_B a \cup_{\mathcal{A}} a'$, then $a' \succsim_C a \cup_{\mathcal{A}} a'$ by Lemma 3a, hence $a \not\bowtie_C a'$ by Lemma 2b, hence $a \succsim_C a'$ by strong consistency, i.e. $a \sim_B a \cup_{\mathcal{A}} a'$ by Lemma 3c.

Sufficiency. Assume $\succsim_B$ satisfies Axioms F-M and F-T. Let $\succsim_C = \succsim_B \setminus \|_B$. Then by Theorem 2, $\succsim_C$ is a cognitive preference relation satisfying Condition F-

LA and $(\succsim_B, \succsim_C)$ is a strongly consistent BC-preference system. Hence by Lemma 3c, it is sufficient to prove that $\succsim_C = \Vdash_B$. Let $a, a' \in \mathcal{A}$. Since $\succsim_B$ is complete, $a \succsim_C a' \Leftrightarrow [a \succsim_B a'$ and $[a \succsim_B a \cup_{\mathcal{A}} a'$ or $a' \succsim_B a \cup_{\mathcal{A}} a']]$. By Axiom F-T, this is equivalent to $[a \succsim_B a'$ and $a \succsim_B a \cup_{\mathcal{A}} a']$. By Axioms F-M and F-T, $a \succsim_B a \cup_{\mathcal{A}} a' \Rightarrow a \succsim_B a \cup_{\mathcal{A}} a' \succsim_B a' \Rightarrow a \succsim_B a'$, hence $a \succsim_C a' \Leftrightarrow a \succsim_B a \cup_{\mathcal{A}} a'$, i.e. $a \Vdash_B a'$ by Axiom F-M. □

The following lemma is needed for the proof of Theorem 4.

Lemma 9. *Assume $\mathcal{A}$ satisfies Structural Axiom (F^{DLR}). Let $\dot{P} \in \mathcal{P}(\mathcal{B}_{EU}(\Delta(\mathcal{Z})))$ and $\dot{\succsim} \in co(\dot{P})$. Then $\forall \dot{\mathbf{u}} \in \mathcal{R}_{EU}(\dot{P})$, $co(\dot{\mathbf{u}}(\dot{P})) \cap \mathcal{R}_{EU}(\dot{\succsim}) \neq \emptyset$.*

Proof. Since $\dot{\succsim} \in co(\dot{P})$, $\exists \dot{\mathbf{u}} \in \mathcal{R}_{EU}(\dot{P})$, $\exists \mu \in \Delta(\dot{\mathbf{u}}(\dot{P}))$ such that $\sum_{\dot{u} \in \dot{\mathbf{u}}(\dot{P})} \mu(\dot{u})\dot{u} \in \mathcal{R}_{EU}(\dot{\succsim})$. Let $\dot{\mathbf{u}}' \in \mathcal{R}_{EU}(\dot{P})$. Then $\forall \dot{u}' \in \dot{\mathbf{u}}'(\dot{P})$, $\exists(\alpha(\dot{u}'), \beta(\dot{u}')) \in \mathbb{R}_+^* \times \mathbb{R}$ such that $\dot{u}' = \alpha(\dot{u}')\dot{\mathbf{u}}(\dot{\mathbf{u}}'^{-1}(\dot{u}')) + \beta(\dot{u}')$, where $\dot{\mathbf{u}}'^{-1} \in \mathcal{F}(\dot{\mathbf{u}}'(\dot{P}) \to \dot{P})$ is defined by $\forall(\dot{\succsim}, \dot{u}') \in \dot{P} \times \dot{\mathbf{u}}'(\dot{P})$,

$$\dot{u}' = \dot{\mathbf{u}}'(\dot{\succsim}) \Leftrightarrow \dot{\succsim} = \dot{\mathbf{u}}'^{-1}(\dot{u}')$$

(this is well defined since two different binary relations cannot admit a common utility representation, hence $\dot{\mathbf{u}}' \in \mathcal{R}_{EU}(\dot{P})$ is injective). Define $\mu', \tilde{\mu}' \in \mathcal{F}(\dot{\mathbf{u}}'(\dot{P}) \to \mathbb{R}_+)$ by $\forall \dot{u}' \in \dot{\mathbf{u}}'(\dot{P})$,

$$\mu'(\dot{u}') = \frac{\mu(\dot{\mathbf{u}}(\dot{\mathbf{u}}'^{-1}(\dot{u}')))}{\alpha(\dot{u}')}, \qquad \tilde{\mu}'(\dot{u}') = \frac{\mu'(\dot{u}')}{\sum_{\dot{u}'' \in \dot{\mathbf{u}}'(\dot{P})} \mu'(\dot{u}'')}.$$

Then $\tilde{\mu}' \in \Delta(\dot{\mathbf{u}}'(\dot{P}))$ and $\sum_{\dot{u}' \in \dot{\mathbf{u}}'(\dot{P})} \tilde{\mu}'(\dot{u}')\dot{u}' = \sum_{\dot{u} \in \dot{\mathbf{u}}(\dot{P})} \mu(\dot{u})\dot{u} + \beta$, where $\beta = \sum_{\dot{u}' \in \dot{\mathbf{u}}'(\dot{P})} \tilde{\mu}'(\dot{u}')\beta(\dot{u}')$, hence $\sum_{\dot{u}' \in \dot{\mathbf{u}}'(\dot{P})} \tilde{\mu}'(\dot{u}')\dot{u}' \in \mathcal{R}_{EU}(\dot{\succsim})$. □

Proof of Theorem 4. note that $co(\dot{P}(\dot{U}_1)) = co(\dot{P}(\dot{U}_2)) \Leftrightarrow [\dot{P}(\dot{U}_1) \subseteq co(\dot{P}(\dot{U}_2))$ and $\dot{P}(\dot{U}_2) \subseteq co(\dot{P}(\dot{U}_1))]$. Hence it is sufficient to prove that $\succsim_{C_1} \subseteq \succsim_{C_2}$ if and only if $\dot{P}(\dot{U}_2) \subseteq co(\dot{P}(\dot{U}_1))$.

If. Assume $\dot{P}(\dot{U}_2) \subseteq co(\dot{P}(\dot{U}_1))$. Then by Lemma 9, $\exists \dot{\mathbf{u}}_\mathbf{2} \in \mathcal{R}_{EU}(\dot{P}(\dot{U}_2))$ such that $\dot{\mathbf{u}}_\mathbf{2}(\dot{P}(\dot{U}_2)) \subseteq co(\dot{U}_1)$. Let $a, a' \in \mathcal{A}$ such that $a \succsim_{C_1} a'$. Then by (1), $\forall \dot{u}_1 \in \dot{U}_1$, $\sup_{x \in a} \dot{u}_1(x) \geq \sup_{x' \in a'} \dot{u}_1(x')$. Hence $\forall \dot{\succsim}_2 \in \dot{P}(\dot{U}_2)$, $\sup_{x \in a} \dot{\mathbf{u}}_\mathbf{2}(\dot{\succsim}_2)(x) \geq \sup_{x' \in a'} \dot{\mathbf{u}}_\mathbf{2}(\dot{\succsim}_2)(x')$, since $\dot{\mathbf{u}}_\mathbf{2}(\dot{P}(\dot{U}_2)) \subseteq co(\dot{U}_1)$. Now $\forall \dot{u}_2 \in \dot{U}_2$, $\exists \dot{\succsim}_2 \in \dot{P}(\dot{U}_2)$ such that $\dot{u}_2 \in \mathcal{R}_{EU}(\dot{\succsim}_2)$, i.e. $\exists(\alpha(\dot{u}_2), \beta(\dot{u}_2)) \in \mathbb{R}_+^* \times \mathbb{R}$ such that $\dot{u}_2 = \alpha(\dot{u}_2)\dot{\mathbf{u}}_\mathbf{2}(\dot{\succsim}_2) + \beta(\dot{u}_2)$, hence $\sup_{x \in a} \dot{u}_2(x) \geq \sup_{x' \in a'} \dot{u}_2(x')$. Hence $a \succsim_{C_2} a'$ by (1).

Only if. Assume $\dot{P}(\dot{U}_2) \not\subseteq co(\dot{P}(\dot{U}_1))$. $\forall j \in \{1, 2\}$, let $\dot{\mathbf{u}}_\mathbf{j} \in \mathcal{R}_{EU}(\dot{P}(\dot{U}_j))$ such that $\forall \dot{\succsim}_j \in \dot{P}(\dot{U}_j)$, $\sum_{z \in \mathcal{Z}} \ddot{v}(\dot{\succsim}_j)(z) = 0$, where $\ddot{v}(\dot{\succsim}_j)$ is the Von Neumann-

Morgenstern utility function inducing $\dot{\mathbf{u}}_{\mathbf{j}}(\dot{\succsim}_j)$. Since $\dot{P}(\dot{U}_2) \not\subseteq co(\dot{P}(\dot{U}_1))$, $\exists \dot{\succsim}_2 \in \dot{P}(\dot{U}_2)$ such that $co(\dot{\mathbf{u}}_{\mathbf{1}}(\dot{P}(\dot{U}_1))) \cap \mathcal{R}_{EU}(\dot{\succsim}_2) = \emptyset$, hence $\dot{\mathbf{u}}_{\mathbf{2}}(\dot{\succsim}_2) \notin co(\dot{\mathbf{u}}_{\mathbf{1}}(\dot{P}(\dot{U}_1)))$. Hence $\forall \mu_1 \in \Delta(\dot{P}(\dot{U}_1))$,

$$\sum\nolimits_{\dot{\succsim}_1 \in \dot{P}(\dot{U}_1)} \mu_1(\dot{\succsim}_1)\ddot{v}(\dot{\succsim}_1) \neq \ddot{v}(\dot{\succsim}_2). \tag{5}$$

In order to use Farkas Lemma, I now prove that every $\mu_1 \in \mathcal{F}(\dot{P}(\dot{U}_1) \to \mathbb{R}_+)$ satisfies (5). Suppose some $\mu_1 \in \mathcal{F}(\dot{P})(\dot{U}_1) \to \mathbb{R}_+)$ violates (5). If $\mu_1 = 0$, then $\dot{\mathbf{u}}_{\mathbf{2}}(\dot{\succsim}_2) = 0$, hence $\dot{U}_2$ is not relevant, a contradiction. If $\mu_1 \neq 0$, define $\tilde{\mu}_1 \in \mathcal{F}(\dot{P}(\dot{U}_1) \to \mathbb{R}_+)$ by

$$\tilde{\mu}_1(\dot{\succsim}_1) = \frac{\mu_1(\dot{\succsim}_1)}{\alpha}, \text{ where } \alpha = \sum\nolimits_{\dot{\succsim}'_1 \in \dot{P}(\dot{U}_1)} \mu_1(\dot{\succsim}'_1) \in \mathbb{R}^*_+.$$

Then $\tilde{\mu}_1 \in \Delta(\dot{P}(\dot{U}_1))$, hence $\alpha\dot{\mathbf{u}}_{\mathbf{2}}(\dot{\succsim}_2) \in co(\dot{\mathbf{u}}_{\mathbf{1}}(\dot{P}(\dot{U}_1)))$, a contradiction since $\alpha\dot{\mathbf{u}}_{\mathbf{2}}(\dot{\succsim}_2) \in \mathcal{R}_{EU}(\dot{\succsim}_2)$.

By Farkas Lemma, $\exists \lambda \in \mathcal{F}(\mathcal{Z} \to \mathbb{R})$ such that

$$\begin{cases} \forall \dot{\succsim}_1 \in \dot{P}(\dot{U}_1), \ \sum_{z \in \mathcal{Z}} \lambda(z)\ddot{v}(\dot{\succsim}_1)(z) \geq 0, \\ \sum_{z \in \mathcal{Z}} \lambda(z)\ddot{v}(\dot{\succsim}_2)(z) < 0. \end{cases}$$

Define $\lambda', \tilde{\lambda}' \in \mathcal{F}(\mathcal{Z} \to \mathbb{R})$ by $\forall z \in \mathcal{Z}$,

$$\lambda'(z) = \lambda(z) - \frac{1}{\#\mathcal{Z}} \sum\nolimits_{z' \in \mathcal{Z}} \lambda(z'), \qquad \tilde{\lambda}'(z) = \frac{\lambda'(z)}{\#\mathcal{Z} \max_{z' \in \mathcal{Z}} |\lambda'(z')|},$$

where $|.|$ denotes absolute value. Then

$$\begin{cases} \forall \dot{\succsim}_1 \in \dot{P}(\dot{U}_1), \ \sum_{z \in \mathcal{Z}} \tilde{\lambda}'(z)\ddot{v}(\dot{\succsim}_1)(z) \geq 0, \\ \sum_{z \in \mathcal{Z}} \tilde{\lambda}'(z)\ddot{v}(\dot{\succsim}_2)(z) < 0, \\ \sum_{z \in \mathcal{Z}} \tilde{\lambda}'(z) = 0 \text{ and } \forall z \in \mathcal{Z}, \ \left|\tilde{\lambda}'(z)\right| \leq \dfrac{1}{\#\mathcal{Z}}. \end{cases}$$

Define $x_0, \tilde{x}_0 \in \mathcal{F}(\mathcal{Z} \to \mathbb{R})$ by $\forall z \in \mathcal{Z}$,

$$x_0(z) = \frac{1}{\#\mathcal{Z}}, \qquad \tilde{x}_0(z) = x_0(z) + \tilde{\lambda}'(z).$$

Then $x_0, \tilde{x}_0 \in \Delta(\mathcal{Z})$ and

$$\begin{cases} \forall \dot{\succsim}_1 \in \dot{P}(\dot{U}_1),\ \dot{\mathbf{u}}_\mathbf{1}(\dot{\succsim}_1)(\tilde{x}_0) \geq \dot{\mathbf{u}}_\mathbf{1}(\dot{\succsim}_1)(x_0), \\ \dot{\mathbf{u}}_\mathbf{2}(\dot{\succsim}_2)(\tilde{x}_0) < \dot{\mathbf{u}}_\mathbf{2}(\dot{\succsim}_2)(x_0). \end{cases}$$

Hence by (2), $\{\tilde{x}_0\} \succsim_{C_1} \{x_0\}$ and $\{\tilde{x}_0\} \not\succsim_{C_2} \{x_0\}$. Hence $\succsim_{C_1} \not\subseteq \succsim_{C_2}$. □

Proof of Lemma 4. *a.* $\mathcal{A}_1$ satisfies Structural Axiom F' since any finite union of bounded sets is bounded. To prove that it also satisfies Structural Axiom M, define $\approx \in \mathcal{B}(\mathcal{A}_1)$ by $\forall a, a' \in \mathcal{A}_1$,

$$a \approx a' \Leftrightarrow [\exists \varepsilon \in \mathbb{R} \text{ such that } a = \{(\dot{a}', m' + \varepsilon) : (\dot{a}', m') \in a'\}]. \tag{6}$$

Then $\approx$ is an equivalence relation (i.e. a reflexive, symmetric, and transitive binary relation), hence $\Phi = \{\phi(a) : a \in \mathcal{A}_1\}$, where $\forall a \in \mathcal{A}_1$, $\phi(a) = \{a' \in \mathcal{A}_1 : a \approx a'\}$, is a partition of $\mathcal{A}_1$. The boundedness requirement in (3) implies that given any $(a, a') \in \mathcal{A}_1 \times \mathcal{A}_1$ such that $a \approx a'$, (6) defines a unique $\varepsilon \in \mathbb{R}$; denote it by $\varepsilon(a, a')$. In each $\phi \in \Phi$, select some $a_0 \in \phi$ (using the axiom of choice), and define $m_\phi \in \mathcal{F}(\phi \to \mathbb{R})$ by $\forall a \in \phi$, $m_\phi(a) = \varepsilon(a, a_0)$. Then m_ϕ is bijective (it is injective by definition, and surjective since boundedness of a subset of $\mathbb{R}$ is preserved by addition of $\varepsilon \in \mathbb{R}$ to all of its elements). Define $\geq_\phi \in \mathcal{B}(\phi)$ by $\forall a, \tilde{a} \in \phi$,

$$a \geq_\phi \tilde{a} \Leftrightarrow m_\phi(a) \geq m_\phi(\tilde{a}).$$

Then $(\phi, \geq_\phi)$ is an unbounded and dense ordered space.

b. $\mathcal{A}_2$ satisfies Structural Axiom M since it is a subset of $\mathcal{A}_1$ such that $\forall a, a' \in \mathcal{A}_1$, $[a \in \mathcal{A}_2 \text{ and } a' \in \phi(a)] \Rightarrow a' \in \mathcal{A}_2$. To prove that it also satisfies Structural Axiom F, define the operator $\cup_{\mathcal{A}_2}$ on $\mathcal{A}_2$ by $\forall a, a' \in \mathcal{A}_2$,

$$a \cup_{\mathcal{A}_2} a' = \{(\dot{a}, m) \in a \cup a' : \forall \varepsilon \in \mathbb{R}_+^*,\ (a, m + \varepsilon) \notin a \cup a'\}.$$

Then $\cup_{\mathcal{A}_2}$ is an idempotent, commutative, and associative operator on $\mathcal{A}_2$. □

Proof of Lemma 5. *a.* Let $a, a' \in \mathcal{A}$. If $a \succsim_C a \cup_\mathcal{A} a'$, then $a \uparrow\succ_C a \cup_\mathcal{A} a'$ by Condition M-M, hence $a \uparrow\succ_B a \cup_\mathcal{A} a'$ by weak consistency. Conversely, if $a \uparrow\succ_B a \cup_\mathcal{A} a'$, then $a \not\prec_B a \cup_\mathcal{A} a'$ by M-regularity, i.e. $a \succsim_C a \cup_\mathcal{A} a'$ by F-completeness. One can prove similarly that $a \cup_\mathcal{A} a' \succsim_C a \Leftrightarrow a \cup_\mathcal{A} a' \uparrow\succ_B a$.

b. Whether $\bowtie_C = \|_C$ or $[\bowtie_C = \|_B^*$ and $\|_B^*$ is F-empty$]$, $\succsim_C$ is F-complete, hence $\|_C = \|_B^*$ by a.

c. Whether $\succsim_C = \Vdash_C$ or $[\succsim_C = \Vdash_B^*$ and $\Vdash_B^*$ is F-complete$]$, $\succsim_C$ is F-complete, hence $\Vdash_C = \Vdash_B^*$ by a. □

Proof of Theorem 5. Assume $\succsim_{C_1} \neq \succsim_{C_2}$. Then $\bowtie_{C_1} = \bowtie_{C_2} = \|_B^*$ by Lemma 5b. Hence $\exists a, a' \in \mathcal{A}$ such that $a \succsim_{C_1} a' \succ_{C_2} a$. Hence $a \uparrow\succ_{C_1} a'$ since $\succsim_{C_1}$ satisfies Condition M-M, hence $a \uparrow\succ_B a'$ since $(\succsim_B, \succsim_{C_1})$ is weakly consistent, hence $a \not\prec_{C_2} a'$ since $(\succsim_B, \succsim_{C_2})$ is M-regular, a contradiction. □

Proof of Lemma 6. *a.* Let $\succsim_C$ be a binary relation on $\mathcal{A}$. If $\succsim_C$ satisfies Condition M-SC, then $\uparrow\succ_C \subseteq \succsim_C \subseteq \not\prec_C$, hence $\prec_C \cap \uparrow\succ_C = \emptyset$. Conversely, if $\succsim_C$ is complete and satisfies Condition M-WC, then $\not\succsim_C \cap \uparrow\succ_C = \emptyset$, i.e. $\uparrow\succ_C \subseteq \succsim_C$.

b. Let $(\succsim_B, \succsim_C)$ be a weakly consistent BC-preference system on $\mathcal{A}$. First note that $(\succsim_B, \succsim_C)$ is M-regular if and only if $\prec_C \cap \uparrow\succ_B = \emptyset$. If $(\succsim_B, \succsim_C)$ is M-regular, then $\prec_C \cap \uparrow\succ_C \subseteq \prec_C \cap \uparrow\succ_B = \emptyset$. Conversely, if $\succsim_C$ is complete and satisfies Conditions M-M and M-WC, then $\prec_C \cap \uparrow\succ_B \subseteq \prec_C \cap \uparrow\succsim_C = \prec_C \cap \uparrow\succ_C = \emptyset$. □

Proof of Lemma 7. Assume $\succsim_C$ violates Condition M-WC. Then $\exists a, a' \in \mathcal{A}$ such that $a \prec_C a'$ and $a \uparrow\succ_C a'$. Let O be an open subset of $\phi(a) \times \phi(a')$. Then since $\mathcal{A}$ is $\geq_{\mathcal{A}}$-dense, $\exists \tilde{a} >_{\mathcal{A}} a$ such that $(\tilde{a}, a') \in O$. Hence $\prec_C \cap(\phi(a) \times \phi(a'))$ intersects the closure of $\succ_C \cap(\phi(a) \times \phi(a'))$. Conversely, assume $\succsim_C$ satisfies Condition M-WC. Let $\phi, \phi' \in \Phi$ and $(a, a') \in \phi \times \phi'$ such that $a \prec_C a'$. Then $\exists \tilde{a} >_{\mathcal{A}} a$, $\exists \tilde{a}' <_{\mathcal{A}} a'$ such that $\tilde{a} \not\succ_C \tilde{a}'$, for otherwise one would have $\forall \tilde{a} >_{\mathcal{A}} a$, $\tilde{a} \succ_C\downarrow a'$, i.e. $\tilde{a} \uparrow\succ_C a'$ by Condition M-S, hence $a \uparrow\succ_C a'$ since $\mathcal{A}$ is $\geq_{\mathcal{A}}$-dense, contradicting condition M-WC. Let

$$O = \{(\hat{a}, \hat{a}') \in \phi \times \phi' : \hat{a} <_{\mathcal{A}} \tilde{a} \text{ and } \hat{a}' >_{\mathcal{A}} \tilde{a}'\}.$$

Then by Conditions M-M and M-S, $\forall (\hat{a}, \hat{a}') \in O$, $\hat{a} \not\succ_C \hat{a}'$. Furthermore, O is open and contains (a, a'). Hence $\prec_C \cap(\phi \times \phi')$ does not intersect the closure of $\succ_C \cap(\phi \times \phi')$. One can prove similarly that $\succ_C \cap(\phi \times \phi')$ does not intersect the closure of $\prec_C \cap(\phi \times \phi')$. □

Proof of Theorem 6. *Necessity.* Assume there exists a cognitive preference relation $\succsim_C$ on $\mathcal{A}$ satisfying Conditions F-LA, M-M, and M-S such that $(\succsim_B, \succsim_C)$ is a weakly consistent and M-regular BC-preference system. Then $\succsim_B$ is FM-complete* by Lemma 5b. Note that $\uparrow\succ_B^*=\uparrow(\succ_B\setminus\bowtie_C)$ by Lemma 5b, hence $\uparrow\succ_B^*\subseteq\uparrow\succsim_C$ by weak consistency, hence $\uparrow\succ_B^*\subseteq\uparrow\succ_C$ by Condition M-M, and that $\uparrow\succ_C\subseteq\uparrow(\succ_B\setminus\|_B^*)=\uparrow\succ_B^*$ by weak consistency and Lemma 5b, so $\uparrow\succ_B^*=\uparrow\succ_C$. Since $\succsim_B^*=\succsim_B\setminus\bowtie_C\subseteq\succsim_C$ by Lemma 5b and weak consistency, this implies that $\succsim_C$ satisfies Axiom FM-M*. Furthermore, using Condition M-S, one can prove similarly that $\succ_B^*\downarrow=\succ_C\downarrow$, hence $\succsim_B$ satisfies Axiom FM-S*. Finally, one has $\not\|_B^*\cap(\|_B^*\lceil)=\not\bowtie_C\cap(\bowtie_C\lceil)$ by Lemma 5b, hence $\not\|_B^*\cap(\|_B^*\lceil)\subseteq\succ_C$ by Condition M-M, hence $\not\|_B^*\cap(\|_B^*\lceil)\subseteq\succsim_B\lceil$ by M-regularity, so $\succsim_B$ satisfies axiom FM-R*.

Sufficiency. Assume $\succsim_B$ is FM-complete* and satisfies Axioms FM-M*, FM-S*, and FM-R*. Let $\succsim_C=(\uparrow\succ_B^*)\setminus\|_B^*$. Then $\succsim_C$ is a cognitive preference relation since $\succsim_B$ is FM-complete* and satisfies Axiom FM-M*. To prove that $\succsim_C$ satisfies Conditions M-M and M-S, first note that $\succsim_C\subseteq\uparrow\succ_B^*=(\uparrow(\uparrow\succ_B^*))\cap(\uparrow\not\|_B^*)=\uparrow\succsim_C$ since $\mathcal{A}$ is $\geq_{\mathcal{A}}$-dense. Hence $\succsim_C\cap(\rceil\precsim_C)\subseteq(\uparrow\succ_B^*)\cap(\rceil(\prec_B^*\uparrow))=(\uparrow\succ_B^*)\cap(\rceil(\downarrow\prec_B^*))$ by Axiom FM-S*. This latter set is empty since $\mathcal{A}$ is $\geq_{\mathcal{A}}$-dense, hence $\succsim_C\subseteq(\uparrow\succsim_C)\setminus(\rceil\precsim_C)=\uparrow\succ_C$. It follows that $\uparrow\succ_C=\uparrow\succsim_C=\uparrow\succ_B^*$. Using Axiom FM-S*, one can prove similarly that $\succ_C\downarrow=\succsim_C\downarrow=\succ_B^*\downarrow$, hence $\uparrow\succ_C=\succ_C\downarrow$. To prove that $(\succsim_C,\succsim_B)$ is weakly consistent and M-regular, let $a,a'\in\mathcal{A}$ such that $a\succ_C a'$. Then $a\not\|_B^* a'$. Suppose $a'\succsim_B a$. Then $a'\succsim_B^* a$, hence $a'\uparrow\succ_B^* a$ by Axiom FM-M*, hence $a'\succsim_C a$, a contradiction. Suppose $a'\uparrow\succ_B a$. Then $a'\uparrow\succ_B^* a$ by Axiom FM-R*, same contradiction. Finally, to prove that $\succsim_C$ satisfies Condition F-LA, it is sufficient to prove that $\bowtie_C=\|_B^*$, by Lemma 5b. By definition of $\succsim_C$, one has $\|_B^*\subseteq\bowtie_C$. Conversely, let $a,a'\in\mathcal{A}$ such that $a\not\|_B^* a'$. Then $a\succsim_B a'\Rightarrow a\succsim_B^* a'\Rightarrow a\uparrow\succ_B^* a'$ by Axiom FM-M*, which implies $a\succsim_C a'$. Similarly, $a'\succsim_B a\Rightarrow a'\succsim_C a$. Hence $a\not\bowtie_C a'$ since $\succsim_B$ is complete. $\square$

Proof of Lemma 8. Assume $\succsim_B$ violates Axiom FM-SC*. Then $\exists a,a'\in\mathcal{A}$ such that $a\|_B^* a'$ and $a\uparrow\not\|_B^* a'$. Hence $\not\|_B^*\cap(\phi(a)\times\phi(a'))$ is not closed in $\phi(a)\times\phi(a')$ since $\mathcal{A}$ is $\geq_{\mathcal{A}}$-dense, i.e. $\|_B^*\cap(\phi(a)\times\phi(a'))$ is not open. Conversely, assume $\succsim_B$ satisfies Axiom FM-SC*. Let $\phi,\phi'\in\Phi$ and $(a,a')\in\phi\times\phi'$ such that $a\|_B^* a'$. Then $\exists\tilde{a}>_{\mathcal{A}} a$ such that $\tilde{a}\|_B^* a'$. Moreover, $\exists\hat{a}<_{\mathcal{A}} a$ such that $\hat{a}\|_B^* a'$, for otherwise one would have $a'\succ_B^*\downarrow a$ by Axiom FM-M*, i.e. $a'\uparrow\succ_B^* a$ by Axiom FM-S*, hence $a'\not\|_B^* a$, a contradiction. Hence by Axiom FM-M* $\forall\bar{a}\in\phi$ such that $\hat{a}<_{\mathcal{A}}\bar{a}<_{\mathcal{A}}\tilde{a}$, $\bar{a}\|_B^* a'$. $\forall\bar{a}\in\phi$ such that $\hat{a}<_{\mathcal{A}}\bar{a}<_{\mathcal{A}}\tilde{a}$, one can show similarly that $\exists\tilde{a}'(\bar{a})>_{\mathcal{A}} a'$, $\exists\hat{a}'(\bar{a})<_{\mathcal{A}} a'$ such that $\forall\bar{a}'\in\phi'$ such that $\hat{a}'(\bar{a})<_{\mathcal{A}}\bar{a}'<_{\mathcal{A}}\tilde{a}'(\bar{a})$, $\bar{a}'\|_B^*\bar{a}$. Let

$$O=\{(\bar{a},\bar{a}')\in\phi\times\phi':\hat{a}<_{\mathcal{A}}\bar{a}<_{\mathcal{A}}\tilde{a}\text{ and }\hat{a}'(\bar{a})<_{\mathcal{A}}\bar{a}'<_{\mathcal{A}}\tilde{a}'(\bar{a})\}.$$

Then O is a neighborhood of (a, a') which is contained in $\|_B^* \cap (\phi \times \phi')$. $\square$

Proof of Theorem 7. *Necessity.* Let $\succsim_C$ be a cognitive preference relation on $\mathcal{A}$ satisfying Conditions F-DP, M-M, M-S, and M-SC such that $(\succsim_B, \succsim_C)$ is a weakly consistent and M-regular BC-preference system. Then by Lemma 2b, $\succsim_B$ is FM-complete* and satisfies Axioms FM-M*, M-S*, and FM-R. Furthermore, $\succsim_B$ satisfies Axiom FM-M° by Lemmas 2a and 5a. To prove that $\succsim_B$ satisfies Axiom FM-SC*, let $a, a' \in \mathcal{A}$ such that $a \uparrow\nparallel_B^* a'$, and suppose $a \|_B^* a'$. Then $a \uparrow\not\bowtie_C a'$ and $a \bowtie_C a'$ by Lemmas 2b and 5b. Hence $a \uparrow\succ_C a'$ by Conditions M-M and M-S. Hence $a \succsim_C a'$ by Condition M-SC, a contradiction. To prove that $\succsim_B$ satisfies Axiom FM-T°, let $a, a' \in \mathcal{A}$. If $a \cup_{\mathcal{A}} a' \succsim_B^\circ a \succsim_B^\circ a'$, then $a \cup_{\mathcal{A}} a' \succsim_B^\circ a'$ by Axiom FM-M°. If $a \succsim_B^\circ a \cup_{\mathcal{A}} a' \succsim_B^\circ a'$, then $a \sim_B^\circ a \cup_{\mathcal{A}} a'$ by Axiom FM-M°, i.e. $a \succsim_C a'$ by Lemma 3c, hence $a \uparrow\succ_C a'$ by Condition M-M, hence $a \succsim_B^\circ a'$ by weak consistency. Finally, if $a \succsim_B^\circ a' \succsim_B^\circ a \cup_{\mathcal{A}} a'$, then first, $a' \succsim_C a \cup_{\mathcal{A}} a'$ by Lemma 5a, hence $a \not\bowtie_C a'$ by Lemma 2b, and second, $a' \not\succ_C a$ by M-regularity, hence $a \succsim_C a'$, i.e. $a \sim_B^\circ a \cup_{\mathcal{A}} a'$ by Lemma 5c.

Sufficiency. Assume $\succsim_B$ satisfies Axioms FM-M*, FM-SC*, FM-S*, FM-R*, FM-M°, and FM-T°. Then by Theorem 6, $\succsim_C$ is a cognitive preference relation satisfying Conditions F-LA, M-M, and M-S, and $(\succsim_B, \succsim_C)$ is a weakly consistent and M-regular BC-preference system. Furthermore, $\succsim_C = \uparrow\succ_B^*$ by Axiom FM-SC*, hence $\succsim_C = \uparrow\succsim_C = \uparrow\succ_C$ by Condition M-M, so $\succsim_C$ satisfies Condition M-SC. Hence by Lemma 5c, it is sufficient to prove that $\succsim_C = \Vdash_B^*$, i.e. $\uparrow\succ_B^* = \Vdash_B^*$. Let $a, a' \in \mathcal{A}$. If $a \uparrow\succ_B^* a'$, then $a \nparallel_B^* a'$ by Axiom FM-SC*, i.e. $[a \uparrow\succ_B a \cup_{\mathcal{A}} a'$ or $a' \uparrow\succ_B a \cup_{\mathcal{A}} a']$ since $\succsim_B$ is complete, hence $a \uparrow\succ_B a \cup_{\mathcal{A}} a'$ by Axiom FM-T°, i.e. $a \Vdash_B^* a'$ by Axiom FM-M°. Conversely, if $a \Vdash_B^* a'$, then first, $a \uparrow\succ_B a \cup_{\mathcal{A}} a'$, hence $a \uparrow\succ_B a'$ by Axioms FM-M° and FM-T°, and second, $a \nparallel_B^* a'$, hence $a \uparrow\succ_B^* a'$ by Axiom FM-R*. $\square$

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