Partial utilitarianism*

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Abstract

Mongin (1994) proved a multi-profile refinement of Harsanyi (1955)'s Aggregation Theorem: within the expected utility model, a social welfare functional mapping profiles of individual utility functions into social preference relations satisfies the Pareto and Independence of Irrelevant Alternatives principles if and only if it is utilitarian for some unique and profile-independent vector of individual weights. The present paper extends this multi-profile setting by allowing individuals to have incomplete preferences, represented by sets of utility functions. An impossibility theorem is first established: social preferences cannot satisfy all the expected utility axioms, precluding utilitarian aggregation in this extended setting. Possibility results are then obtained by relaxing either the completeness or the independence axiom at the social level, yielding two forms of *partial* utilitarianism.

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JEL Classification. D71, D81.

1 Introduction

Comparing social alternatives – such as allocations resulting from alternative economic policies – requires choosing a social welfare criterion to evaluate these alternatives. Considering a finite society where each individual agent i is endowed with a utility function u_i, the most widely used criteria are Bentham (1781)'s *utilitarian* criterion $\sum_i u_i$ and Rawls (1971)'s *egalitarian* criterion $\min_i u_i$. Social choice theory, in turn, can guide the choice of a social welfare criterion by providing axiomatic foundations for the various criteria under consideration. In particular, when social alternatives involve risk and both individuals and society conform to von Neumann and Morgenstern (1944)'s expected utility model (vNM's EU model henceforth), Harsanyi (1955) showed that the only social welfare criteria satisfying the Pareto principle with respect to individual preferences are utilitarian criteria of the form $\sum_i \theta_i u_i$ for some vector θ of individual weights – we will henceforth refer to Bentham's special case where all individuals have equal weight as *classical* utilitarianism. This "aggregation theorem" has provided a powerful – though controversial – defense of utilitarianism.[1]

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[1]It is not to be confused with Harsanyi (1953)'s "impartial observer" theorem," which provides distinct axiomatic foundations for utilitarianism.

In many situations, however, individual preferences may well be incomplete – i.e. leave some alternatives mutually unranked – and, hence, fail to be representable by a utility function. We may consider, for instance, the decision of whether or not to get vaccinated against a new disease when both the disease and vaccine have largely unknown consequences. The importance of allowing for incomplete preferences to model individual indecisiveness was highlighted by von Neumann and Morgenstern (1944, p. 19), Aumann (1962, p. 446), and Schmeidler (1989, p. 576), and representation theorems relaxing the Completeness axiom have been established in various settings (Bewley, 1986; Shapley and Baucells, 1998; Ok, 2002; Dubra et al., 2004; Ok et al., 2012; Galaabaatar and Karni, 2013; Riella, 2015). Incomplete preferences also arise naturally – without being taken as primitive – in models when an individual can have uncertain tastes (Koopmans, 1964; Kreps, 1979; Dekel et al., 2001) or be influenced by multiple "selves", "rationales", "frames", or "ancillary conditions" (May, 1954; Kalai et al., 2002; Salant and Rubinstein, 2008; Bernheim and Rangel, 2009; Ambrus and Rozen, 2014). In a social choice setting, incompleteness may also reflect partial identification of individual preferences by the social planner (Manski, 2005, 2010, 2013).

The present paper analyses the aggregation of potentially incomplete EU preferences over risky alternatives. Our starting point is a reformulation of Harsanyi's aggregation theorem due to Mongin (1994), casting Harsanyi's result into Sen (1970)'s *social welfare functional* (SWFL) setting.[2] That is, whereas Harsanyi considers a single profile (u_i) of individual vNM utility functions and a social EU preference relation, Mongin considers a SWFL associating a social EU preference relation to each conceivable profile (u_i) of vNM individual utility functions.[3] Adding to the Pareto principle an Independence of Irrelevant Alternatives (IIA) principle that is common in this multi-profile setting, Mongin obtains a characterization of the utilitarian criteria $\sum_i \theta_i u_i$, with the additional benefit over Harsanyi's single-profile result that the weight vector θ is uniquely determined and independent of the utility profile.[4] This allows, in particular, to characterize Bentham's classical utilitarianism – through an additional Anonymity axiom.

To allow for individual incompleteness in Mongin's theorem, we consider an *extended social welfare functional* (ESWFL) associating a social EU preference relation to each conceivable profile (U_i) of individual *sets* of vNM utility functions, viewing such sets as "multi-utility" representations of incomplete EU preferences (Shapley and Baucells, 1998; Dubra et al., 2004). An impossibility theorem is first established in this setting: under the Pareto and IIA principles, social preferences cannot systematically satisfy all the EU axioms, unless they trivially boil down to full indifference – this impossibility holds even without requiring social preferences to satisfy the Mixture Continuity axiom. Hence an ESWFL satisfying these two principles cannot be utilitarian in the sense that for each profile (U_i), the corresponding social preferences admit a representation of the form $\sum_i \theta_i u_i$ for some $u_i \in U_i$.

Possibility results are then obtained, which overcome this impossibility by relaxing the EU axioms at the social level. On the one hand, relaxing the Completeness axiom yields a representation of *coherent* social preferences by a set of utility functions of the form $\{\sum_i \theta_i u_i : \theta \in \Theta^*, u_i \in U_i\}$ for some set Θ^* of weight vectors. On the other hand, relaxing Independence yields a representation of *decisive* social preferences by a utility function of the form $\min_{\theta \in \Theta^\wedge} \sum_i \theta_i \min_{u_i \in U_i} u_i(\cdot)$ for some set $\Theta^\wedge$ of weight

[2] For a survey of the SWFL literature, see d'Aspremont and Gevers (2002).

[3] Although Harsanyi's theorem is sometimes viewed as taking preferences as the only primitives, the weight vector θ is essentially arbitrary unless utility representations of individual preferences are fixed.

[4] In Harsanyi's theorem, in contrast, a necessary – but not sufficient – condition for θ to be uniquely determined by the profile (u_i) is that there be no more individuals than pure outcomes.

vectors. Finally, axioms connecting coherent and decisive social preferences ensure that $\Theta^* = \Theta^\wedge$. These results bear formal similarities with the "objective-subjective rationality" model of Gilboa et al. (2010) in the context of individual decision making under uncertainty, although particularities arise in the present social choice setting.

Coherent and decisive rational social preferences are thus both fully determined by a set Θ of weight vectors, which is unique and independent of the profile (U_i). They are *partially* utilitarian in the sense that they more precisely rely on the set of all utilitarian criteria of the form $\sum_i \theta_i u_i$ where $\theta \in \Theta$ and $u_i \in U_i$. Coherent social preferences correspond to unanimity across all these criteria. The larger Θ, the more social incompleteness. When Θ is maximal, the social set of utility functions boils down to $\bigcup_i U_i$ and social preferences reduce to the Pareto dominance relation. As for decisive social preferences, each social alternative is evaluated by means of the least favorable of these criteria. The larger Θ, the more social violations of the Independence axiom. When Θ is maximal, the social utility function boils down to $\min_i \min_{u_i \in U_i} u_i(\cdot)$ and social preferences correspond to Rawls' egalitarian criterion, extended to also minimize over U_i.

The distinction between coherent and decisive social preferences gives rise to a new interpretation of Diamond (1967)'s critique of Harsanyi's aggregation theorem. It also makes it possible to characterize a more general Hurwicz (1951)-type of representation for decisive social preferences, able to accommodate milder degrees of inequality aversion or even inequality seeking. Similar characterizations were established by Ghirardato et al. (2004) and Frick et al. (2020) in an individual decision making context, although the present social choice setting again yields some specific features.

Harsanyi showed that a weakening of the Pareto principle, the Pareto Indifference principle, is in fact sufficient to characterize utilitarianism, provided individual weights are allowed to be negative. In the present setting, the Pareto Indifference principle suffices for the impossibility theorem. The impossibility in fact persists under a weaker IIA principle, covering in particular the "relative utilitarianism" social welfare criterion (Dhillon, 1998; Dhillon and Mertens, 1999). Generalizations of the two representation theorems under the Pareto Indifference principle are also established. The weight vectors in these representations feature two weights per individual, one positive and one negative, rather than a single weight of arbitrary sign. Besides being more general, the results under Pareto Indifference are mathematically the most substantial results of the paper and require new proof methods, the results under the standard Pareto principle then following as simple corollaries.

Like the present paper, Danan et al. (2013) analyze the aggregation of sets of utility functions in a multi-profile EU setting. The main result obtained there has a similar flavor to the first representation result presented here and also implies an impossibility of utilitarian aggregation. An important difference, however, is that society is endowed there with a set of vNM utility functions rather than a preference relation, making for a stronger IIA axiom. As a consequence, the results are independent of each other and their proofs largely differ. The present setting is a more standard one in the social choice literature. It is also less restrictive in that it allows to relax the Independence and Mixture Continuity axioms at the social level, which the second representation theorem and the impossibility result presented here do, respectively. Also related is Danan et al. (2015)'s generalization of Harsanyi's Aggregation Theorem relaxing the Completeness axiom. In the single-profile setting adopted there, incompleteness does not preclude utilitarian aggregation, but non-uniqueness of the weight vector set is even more severe than in Harsanyi's result. Other social choice theoretic works relaxing the Completeness or Independence axioms in various settings include Gajdos et al. (2008); Crès et al. (2011); Pivato (2011, 2013, 2014); Nascimento (2012); Gajdos and Vergnaud (2013); Chambers and Hayashi (2014); Qu (2015); Danan et al. (2016); Alon and Gayer (2016); Zuber (2016); McCarthy et al. (2019, 2020).

The paper is organized as follows. Section 2 introduces the formal setup. Section 3 reviews Mongin's theorem. Section 4 contains the impossibility theorem. Sections 5–7 present the representation theorems. Section 8 analyzes some special cases. Section 9 discusses the issue of interpersonal utility comparisons. Section 10 concludes. All proofs appear in the Appendix.

2 Alternatives, preferences, utility

Let X be a set of social alternatives and assume that X is a convex subset of some linear space and that X contains at least 3 affinely independent alternatives – i.e. the affine dimension of X is at least 2. This is the case, in particular, if X is the set of all lotteries on a set of at least 3 pure outcomes.

A *preference relation* $\succsim$ on X is a binary relation on X, where $x \succsim y$ is interpreted as alternative x being weakly preferred to alternative y. As usual, the symmetric (indifference) and asymmetric (strict preference) components of a preference relation $\succsim$ on X are denoted by $\sim$ and $\succ$, respectively. The following are standard properties of preference relations.

Reflexivity For all $x \in X$, $x \succsim x$.

Completeness For all $x, y \in X$, either $x \succsim y$ or $y \succsim x$.

Transitivity For all $x, y, z \in X$, if $x \succsim y \succsim z$ then $x \succsim z$.

Independence For all $x, y, z \in X$ and all $\lambda \in (0, 1)$, $x \succsim y$ if and only if $\lambda x + (1-\lambda)z \succsim \lambda y + (1-\lambda)z$.

Mixture Continuity For all $x, y, z \in X$, the sets $\{\lambda \in [0, 1] : x \succsim \lambda y + (1-\lambda)z\}$ and $\{\lambda \in [0, 1] : \lambda y + (1-\lambda)z \succsim x\}$ are closed.

A *preorder* (resp. *weak order*) is a reflexive (resp. complete) and transitive preference relation. An *expected utility (EU) preorder* (resp. *weak order*) is a preorder (resp. weak order) satisfying Independence and Mixture Continuity.

A *utility function* u on X associates to each alternative $x \in X$ a utility level $u(x) \in \mathbb{R}$. A utility function u on X is a *von Neunamm-Morgenstern (vNM) utility function* if $u(\lambda x + (1-\lambda)y) = \lambda u(x) + (1-\lambda)u(y)$ for all $x, y \in X$ and all $\lambda \in (0, 1)$. A *utility set* on X is a non-empty set of utility functions on X. A *vNM utility set* on X is a non-empty, compact, and convex subset of P, where P is endowed with the subspace topology and $\mathbb{R}^X$ with the product topology. Let $P \subset \mathbb{R}^X$ denote the set of all vNM utility functions on X and let $\mathcal{P}$ denote the set of all vNM utility sets on X. P is a linear subspace of $\mathbb{R}^X$ and contains in particular all constant functions, whereas $\mathcal{P}$ contains in particular all convex hulls of finite sets of vNM utility functions on X and, hence, all singletons.

A utility set U on X *represents* a preference relation $\succsim$ on X if for all $x, y \in X$,

$$x \succsim y \Leftrightarrow [\forall u \in U, u(x) \geq u(y)] .$$

When U is a singleton, we simply say as usual that the corresponding utility function represents $\succsim$. A preference relation $\succsim$ on X can be represented by some vNM utility function u on X if and only if $\succsim$ is an EU weak order (von Neumann and Morgenstern, 1944; Herstein and Milnor, 1953). If X is finite-dimensional, a preference relation $\succsim$ on X can more generally be represented by some vNM utility set U on X if and only if $\succsim$ is an EU preorder (Shapley and Baucells, 1998; Dubra et al., 2004).[5] If X is infinite-dimensional, $\succsim$ being an EU preorder is necessary but generally not sufficient for such a representation to exist.[6]

3 Social welfare functionals and utilitarianism

The starting point of our analysis is Mongin (1994)'s multi-profile version of Harsanyi (1955)'s Aggregation Theorem, which we briefly review here. Let I be a non-empty and finite set of individuals and let $\Delta_I = \{\theta \in \mathbb{R}^I_+ : \sum_{i\in I} \theta_i = 1\}$ denote the unit simplex of $\mathbb{R}^I$. Following Sen (1970), a *social welfare functional (SWFL)* f on X associates to each profile $(u_i)_{i\in I} \in P^I$ of individual vNM utility functions on X a social EU weak order $f((u_i)_{i\in I})$ on X, which we also denote by $\succsim_{(u_i)_{i\in I}}$. The following are standard properties of SWFLs.

Pareto Preference For all $(u_i)_{i\in I} \in P^I$ and all $x, y \in X$, if $u_i(x) \geq u_i(y)$ for all $i \in I$ then $x \succsim_{(u_i)_{i\in I}} y$.

Independence of Irrelevant Alternatives (IIA) For all $(u_i)_{i\in I}, (v_i)_{i\in I} \in P^I$ and all $x, y \in X$ such that $u_i(x) = v_i(x)$ and $u_i(y) = v_i(y)$ for all $i \in I$, $x \succsim_{(u_i)_{i\in I}} y$ if and only if $x \succsim_{(v_i)_{i\in I}} y$.

Viewing each individual utility function as representing an underlying preference relation, Pareto Preference requires the social preference relation to preserve all unanimous individual weak preferences. IIA, on the other hand, requires the social ranking between two alternatives to be determined solely by the individual utility levels of these two alternatives, independently of those of any other alternative.

Theorem (Mongin, 1994)**.** A SWFL f on X satisfies Pareto Preference and IIA if and only if there exists a vector $\theta \in \mathbb{R}^I_+$ such that for all $(u_i)_{i\in I} \in P^I$, the vNM utility function

$$u_{\theta,(u_i)_{i\in I}} = \sum_{i\in I} \theta_i u_i$$

represents $\succsim_{(u_i)_{i\in I}}$. Moreover, another vector $\theta' \in \mathbb{R}^I$ represents f as above if and only if $\theta' = \mu\theta$ for some $\mu \in \mathbb{R}_{++}$.

Note that IIA is a "between-profile" property linking social preferences in different profiles whereas Pareto Preference is a "within-profile" property that can be stated for one given profile. By adopting a multi-profile setting and adding the IIA principle, Mongin thus obtained the same utilitarian characterization as Harsanyi, with the additional benefit that the weight vector θ is unique and independent of the utility profile considered – uniqueness is up to a positive scale factor, but it is a simple matter of normalization to make θ fully unique.

[5]Moreover, another vNM utility set V also represents $\succsim$ if and only if the closure of the cone generated by V and the constant functions in $\mathbb{R}^X$ is identical to the closure of the cone generated by U and the constant functions in $\mathbb{R}^X$. This generalizes the standard uniqueness of vNM utility functions up to positive affine transformations.

[6]More precisely, if the dimension of X is countable, being an EU preorder is necessary and sufficient for such a representation where U is closed (but not necessarily compact) to exist (McCarthy et al., 2021). If the dimension of X is uncountable, a "countable domination" property is sufficient but not necessary for such a representation where U is closed to exist (McCarthy et al., 2021). Alternatively, if X is the set of all Borel probability measures on some infinite compact metric space, a stronger continuity property is necessary and sufficient for such a representation where U is closed and each $u \in U$ is continuous with respect to the topology of weak convergence to exist (Dubra et al., 2004).

4 Extended social welfare functionals and impossibility of utilitarianism

We now extend the SWFL setting to allow for individual preference incompleteness and establish an impossibility result in this extended setting. An *extended social welfare functional (ESWFL) F* on X associates to each profile $(U_i)_{i\in I} \in \mathcal{P}^I$ of individual vNM utility sets on X a social preorder $F((U_i)_{i\in I})$ on X, which we also denote by $\succsim_{(U_i)_{i\in I}}$. Note that this allows social preferences to violate the Completeness, Independence, and Mixture Continuity properties that are satisfied in Mongin (1994)'s – and Harsanyi (1955)'s – result, so we explicitly state these properties as axioms.

Axiom (Completeness)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$, $\succsim_{(U_i)_{i\in I}}$ is complete.

Axiom (Independence)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$, $\succsim_{(U_i)_{i\in I}}$ satisfies Independence.

Axiom (Mixture Continuity)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$, $\succsim_{(U_i)_{i\in I}}$ is mixture continuous.

The Pareto and IIA principles need to be generalized in this extended setting. The generalization of the Pareto Preference principle is straightforward.

Axiom (Pareto Preference)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$, if $u_i(x) \geq u_i(y)$ for all $u_i \in U_i$ and all $i \in I$ then $x \succsim_{(U_i)_{i\in I}} y$.

To generalize the IIA principle, we introduce the following notation. Given a subset Y of X and a utility set $U \in \mathcal{P}$, let $U|_Y = \{(u(x))_{x\in Y} : u \in U\} \subset \mathbb{R}^Y$ denote the *restriction* of U to Y.

Axiom (IIA)**.** For all $(U_i)_{i\in I}, (V_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$ such that $U_i|_{\{x,y\}} = V_i|_{\{x,y\}}$ for all $i \in I$, $x \succsim_{(U_i)_{i\in I}} y$ if and only if $x \succsim_{(V_i)_{i\in I}} y$.

Note that for a SWFL, IIA implies that the restriction of the social preference relation to any subset of alternatives depends only on the restrictions of the individual utility functions to this subset Blau (1971); D'Aspremont and Gevers (1977). This is simply because any function is fully determined by its restrictions to all pairs of elements in its domain. This argument does not hold for utility sets, however: a set of functions on a common domain is generally not fully determined by the corresponding sets of restrictions to all pairs of elements of the domain, because there is generally more than one way of "gluing" these sets of restrictions together. Nevertheless, because we restrict attention to vNM utility sets, it can be shown that IIA still implies that the restriction of the social preference relation to any subset of alternatives depends only on the restrictions of the individual utility sets to this subset in the present setting (see Danan et al., 2013, Lemmas 15 and 16).

A final axiom is needed for our impossibility result, which prevents the ESWFL from being trivial in the sense that all alternatives are mutually indifferent in all profiles.

Axiom (Non-Triviality)**.** There exist $(U_i)_{i\in I} \in \mathcal{P}^I$ and $x, y \in X$ such that $x \not\sim_{(U_i)_{i\in I}} y$.

Theorem 1. There exists no ESWFL on X satisfying Pareto Preference, IIA, Completeness, Independence, and Non-Triviality.

Thus, under the Pareto and IIA principles, social preferences cannot systematically and simultaneously satisfy Completeness and Independence – two of the EU axioms – in our extended setting. This is unlike in Mongin's – and Harsanyi's – theorem and, as a consequence, an ESWFL satisfying these two principles cannot be utilitarian in the sense that, for all profile $(U_i)_{i\in I} \in \mathcal{P}^I$, $\succsim_{(U_i)_{i\in I}}$ can be represented by a utility function of the form $\sum_{i\in I} \theta_i u_i$ where $\theta \in \mathbb{R}^I$ and $u_i \in U_i$ for all $i \in I$.

To obtain possibility results, it is therefore necessary to relax one of these two aggregation principles or one of these two EU axioms. Pareto and IIA, however, are fundamental principles of the the multi-profile approach to preference aggregation: without the former, social preferences can hardly be considered as aggregating individual preferences; without the latter, one is essentially brought back to the single-profile setting – where the impossibility disappears but non-uniqueness is pervasive (Danan et al., 2015). Moreover, the impossibility of Theorem 1 persists under arguably minimal versions of these principles – see Theorem 7. We will therefore maintain them throughout and focus on the tension between Completeness and Independence.

5 Coherence vs. decisiveness and partial utilitarianism

In this section we present characterization results circumventing the impossibility of Theorem 1 by relaxing Completeness or Independence – and at the same time imposing Mixture Continuity and Non-Triviality to obtain well-behaved representations. The appeal of Completeness and Independence stems from two distinct goals: whereas the former enables social preferences to guide every possible decision to be made, the latter ensures that social preferences provide a coherent guidance. In view of the incompatibility between these two goals, it seems natural for a social planner to first seek to rely on a coherent guidance and, when it is indecisive, fall back to a fully decisive but less coherent guidance. We therefore consider two ESWFLs, a *coherent* one $F^* = \succsim^*_{(\cdot)}$ satisfying Independence – but not Completeness – and a *decisive* one $F^\wedge = \succsim^\wedge_{(\cdot)}$ satisfying Completeness – but not Independence. This is formally similar to the distinction between “objective” and “subjective” rationality approach put forward by Gilboa et al. (2010) in the context of individual decision making – although the incompatibility between Completeness and Independence only arises in the present social context.

We start with a characterization result for the coherent ESWFL F^*. Relaxing Completeness allows to avoid the impossibility of Theorem 1 and delivers the following representation.

Theorem 2. An ESWFL F^* on X satisfies Pareto Preference, IIA, Independence, Mixture Continuity, and Non-Triviality if and only if there exists a non-empty, compact, and convex set $\Theta^* \subseteq \Delta_I$ such that for all $(U_i)_{i\in I} \in \mathcal{P}^I$, the vNM utility set

$$U_{\Theta,(U_i)_{i\in I}} = \left\{ \sum_{i\in I} \theta_i u_i \,:\, \theta \in \Theta^*, (u_i)_{i\in I} \in \prod_{i\in I} U_i \right\}$$

represents $\succsim^*_{(U_i)_{i\in I}}$. Moreover, Θ^* is unique.

The coherent ESWFLs characterized in Theorem 2 are partially utilitarian in the sense that social preference corresponds to unanimity across a set of utilitarian criteria. The larger this set, the more incomplete social preferences. At one extreme, when $\Theta^* = \Delta_I$, social preferences boil down to the Pareto dominance relation. At the other extreme, when Θ^* is a singleton, social preferences are complete for profiles of singleton utility sets but, consistently with Theorem 1, are necessarily incomplete for other profiles.

We now turn to the decisive ESWFL $F^\wedge$. Although $F^\wedge$ cannot be fully coherent – in the form of Independence – by Theorem 1, it can still achieve a weaker form of coherence. To state the corresponding axiom, say that an alternative $x \in X$ is *egalitarian* in a profile $(U_i)_{i\in I} \in \mathcal{P}^I$ if $u_i(x) = u_j(x)$ for all $i, j \in I$ and all $u_i \in U_i, u_j \in U_j$. Let $\hat{X}_{(U_i)_{i\in I}} \subseteq X$ denote the set of all egalitarian alternatives in $(U_i)_{i\in I}$.

Axiom (Egalitarian Independence)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$, all $x, y \in X$, all $z \in \hat{X}_{(U_i)_{i\in I}}$, and all $\lambda \in (0,1)$, $x \succsim_{(U_i)_{i\in I}} y$ if and only if $\lambda x + (1-\lambda)z \succsim_{(U_i)_{i\in I}} \lambda y + (1-\lambda)z$.

Egalitarian Independence only requires social preferences to be coherent when mixing with an egalitarian alternative. A particularity of egalitarian alternatives is that mixing with them does not affect inequalities in utility levels between individuals. We also impose the following axiom on $F^\wedge$, requiring that a half-half mixture of two indifferent alternatives be weakly preferred to either of them. This is another weakening of Independence and, to the extent that such mixtures reduce inequalities, seems a plausible requirement for a fairness concerned social planner, although perhaps more questionable than Egalitarian Independence. It will be relaxed in Section 6.

Axiom (Inequality Aversion)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$, if $x \sim_{(U_i)_{i\in I}} y$ then $0.5x + 0.5y \succsim_{(U_i)_{i\in I}} y$.

Theorem 3. An ESWFL $F^\wedge$ on X satisfies Pareto Preference, IIA, Completeness, Egalitarian Independence, Inequality Aversion, Mixture Continuity, and Non-Triviality if and only if there exists a non-empty, compact, and convex set $\Theta^\wedge \subseteq \Delta_I$ such that for all $(U_i)_{i\in I} \in \mathcal{P}^I$, the utility function

$$u_{\Theta^\wedge,(U_i)_{i\in I}} : x \mapsto \min_{\theta\in\Theta^\wedge} \sum_{i\in I} \theta_i \min_{u_i\in U_i} u_i(x),$$

represents $\succsim_{(U_i)_{i\in I}}$. Moreover, $\Theta^\wedge$ is unique.

The subjectively rational ESWFLs characterized in Theorem 3 are partially utilitarian in the sense that each alternative is socially evaluated by means of the least favorable of a set of utilitarian criteria. The larger this set, the more social preferences violate Independence. At one extreme, when $\Theta^\wedge = \Delta_I$, social preferences boil down to Rawls (1971)' egalitarian criterion. At the other extreme, when $\Theta^\wedge$ is a singleton, social preferences satisfy Independence for profiles of singleton utility sets but, consistently with Theorem 1, necessarily violate it for other profiles.

Egalitarian Independence and Inequality Aversion are formally similar to the Certainty Independence and Uncertainty Aversion axioms of Gilboa and Schmeidler (1989)'s maxmin EU model in the context of individual decision making under uncertainty. A particularity of the maxmin representation of Theorem 3 with respect to theirs is that minimization is performed simultaneously over weight vectors and utility functions.

Finally, the two following axioms connect the coherent and decisive ESWFLs.

Axiom (Consistency)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$, if $x \succsim^*_{(U_i)_{i\in I}} y$ then $x \succsim^\wedge_{(U_i)_{i\in I}} y$.

Axiom (Egalitarian Default)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$, all $x \in X$, and all $y \in \hat{X}_{(U_i)_{i\in I}}$, if $x \not\succsim^*_{(U_i)_{i\in I}} y$ then $y \succsim^\wedge_{(U_i)_{i\in I}} x$.

Consistency prevents decisive preferences form overturning coherent preferences. This reflects our interpretation of the former as completing the latter when they are indecisive. Egalitarian Default requires egalitarian alternatives to be systematically favored in the absence of a coherent preference. Like

Inequality Aversion, this is plausible for a fairness concerned social planner but not unquestionable, and will be relaxed in Section 6. Consistency and Egalitarian Default are analogues in the present setting to the Consistency and Caution axioms of Gilboa et al. (2010). They deliver the following joint representation.

Theorem 4. The following are equivalent for a pair of ESWFLs $(F^*, F^\wedge)$:

(i) F^* satisfies Pareto Preference, IIA, Independence, Mixture-Continuity, and Non-Triviality; $F^\wedge$ satisfies IIA, Completeness, Egalitarian Independence, Mixture Continuity, and Non-Triviality; and jointly $(F^*, F^\wedge)$ satisfy Consistency and Egalitarian Default.

(ii) There exists a non-empty, compact, and convex set $\Theta \subseteq \Delta_I$ representing F^* as per Theorem 2 and $F^\wedge$ as per Theorem 3.

Moreover, Θ is unique.

Note that it is not necessary to assume that $F^\wedge$ satisfies Pareto Preference and Inequality Aversion, as these are implied by the other axioms. Also, if Egalitarian Default is strengthened by requiring that $y \succ^\wedge_{(U_i)_{i\in I}} x$ (similarly to Gilboa et al. (2010)'s Default to Certainty axiom), then it is not necessary to assume that $F^\wedge$ satisfies Egalitarian Independence either.

Theorem 4 yields a new interpretation of Diamond (1967)'s critique of Harsanyi's Aggregation Theorem. Let $I = \{1, 2\}$ and consider a profile $(\{u_1\}, \{u_2\})$ of singleton utility sets as well as two alternatives $x, y \in X$ such that $u_1(x) = u_2(y) = 1$ and $u_1(y) = u_2(x) = 0$. Diamond (1967) argued that a social planner indifferent between the alternatives x and y might nevertheless prefer the egalitarian alternative $0.5x + 0.5y$ to them. Within the framework of Theorem 4, this preference pattern can only live in the decisive ESWFL – it violates Independence but not Egalitarian Independence. In the objective ESWFL, on the other hand, x, y, and $0.5x + 0.5y$ must be mutually unranked.

6 Inequality attitudes

The representation of the decisive ESWFL in Theorems 3 and 4 may seem quite restrictive in that it focuses exclusively on the least favorable weight vector and utility functions – although it should be noted that the relevant set of weight vectors is part of the representation. In this section we obtain a more general α-maxmin representation of the decisive ESWFL in the spirit of Hurwicz (1951), allowing for milder degrees of inequality aversion or even inequality seeking.

We first establish a generalization of Theorem 3 doing away with the Inequality Aversion axiom. To this end, let $2I = I \sqcup I$, where $\sqcup$ denotes disjoint union, stand for a population made of two copies of each individual $i \in I$. Let $D = \{(s, t) \in \mathbb{R}^{2I} : s \le t\}$ and say that a functional $h : D \to \mathbb{R}$ is:

- *monotonic* if $h(s, t) \ge h(s', t')$ for all $(s, t), (s', t') \in D$ such that $s \ge s'$ and $t \ge t'$,
- *positively homogeneous* if $h(\mu(s, t)) = \mu h(s, t)$ for all $(s, t) \in D$ and all $\mu \in \mathbb{R}_+$,
- *constant additive* if $h((s, t) + c) = h(s, t) + c$ for all $(s, t) \in D$ and all $c \in \mathbb{R}$,
- *constant linear* if it is positively homogeneous and constant additive.

Theorem 5. An ESWFL $F^\wedge$ on X satisfies Pareto Preference, IIA, Completeness, Egalitarian Independence, Mixture Continuity, and Non-Triviality if and only if there exists a monotonic and constant linear functional $h : D \to \mathbb{R}$ such that for all $(U_i)_{i\in I} \in \mathcal{P}^I$, the utility function

$$u_{h,(U_i)_{i\in I}} : x \mapsto h\left(\left(\min_{u_i\in U_i} u_i(x)\right)_{i\in I}, \left(\max_{u_i\in U_i} u_i(x)\right)_{i\in I}\right)$$

represents $\succsim^\wedge_{(U_i)_{i\in I}}$. Moreover, h is unique.

The representation in Theorem 5 is in the spirit of Ghirardato et al. (2004)'s representation of invariant biseparable preferences in the context of individual decision making under uncertainty. A particular feature arising in the present setting is that the functional h can only depend on the minimal and maximal utility levels of all individuals for the alternative under consideration.

We now weaken the Egalitarian Default axiom in Theorem 4 to obtain an α-maxmin representation of the decisive ESWFL. In the absence of a coherent social ranking between two alternatives x and y, a natural way to try and decide between them is to compare how they are coherently ranked with respect to egalitarian alternatives.[7] The following axiom requires the social planner to decide on x rather than y whenever x clearly dominates y in the sense that, according to the coherent social ranking, every egalitarian alternative above x is also above y while every alternative below y is also below x. Formally given a profile $(U_i)_{i\in I} \in \mathcal{P}^I$ and two alternatives $x, y \in X$, say that x *egalitarian dominates* y, denoted $x \trianglerighteq^*_{(U_i)_{i\in I}} y$, if for all $(V_i)_{i\in I} \in \mathcal{P}^I$ such that $U_i|_{\{x,y\}} = V_i|_{\{x,y\}}$ for all $i \in I$ and all $z \in \hat{X}_{(V_i)_{i\in I}}$, $z \succsim^*_{(V_i)_{i\in I}} x$ implies $z \succsim^*_{(V_i)_{i\in I}} y$ and $y \succsim^*_{(V_i)_{i\in I}} z$ implies $x \succsim^*_{(V_i)_{i\in I}} z$.

Axiom (Egalitarian Dominance)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$, if $x \trianglerighteq^*_{(U_i)_{i\in I}} y$ then $x \succsim^\wedge_{(U_i)_{i\in I}} y$.

Under the assumptions of Theorem 2, Egalitarian Dominance implies Consistency but is weaker than the conjunction of Consistency and Egalitarian Default. It delivers the following representation.

Theorem 6. The following are equivalent for a pair of ESWFLs $(F^*, F^\wedge)$:

(i) F^* satisfies Pareto Preference, IIA, Independence, Mixture Continuity, and Non-Triviality; $F^\wedge$ satisfies IIA, Completeness, Egalitarian Independence, Mixture Continuity, and Non-Triviality; and jointly $(F^*, F^\wedge)$ satisfy Egalitarian Dominance.

(ii) There exists a non-empty, compact, and convex set $\Theta \subseteq \Delta_I$ representing F^* as per Theorem 2 and a constant $\alpha \in [0, 1]$ such that for all $(U_i)_{i\in I} \in \mathcal{P}^I$, the utility function

$$u_{\Theta,\alpha,(U_i)_{i\in I}} : x \mapsto \alpha \min_{\theta\in\Theta} \sum_{i\in I} \theta_i \min_{u_i\in U_i} u_i(x) + (1-\alpha) \max_{\theta\in\Theta} \sum_{i\in I} \theta_i \max_{u_i\in U_i} u_i(x)$$

represents $\succsim^\wedge_{(U_i)_{i\in I}}$.

Moreover, Θ and α are unique.

Egalitarian Dominance is formally similar to Frick et al. (2020)'s "Security Potential Dominance" axiom in the context of individual decision making under uncertainty.[8] Two particularities of the α-maxmin representation of Theorem 6 with respect to theirs are that (i) minimization and maximization

[7] Besides being particularly simple and well-behaved with respect to mixing by Egalitarian Independence, these alternatives are completely ordered by Pareto Preference.

[8] Theorem 6 could alternatively be stated by adding to Consistency an axiom similar to Ghirardato et al. (2004)'s Axiom 7.

are performed simultaneously over weight vectors and utility functions and (ii) the constant α measuring the social planner's inequality aversion is unique even when there is a single weight vector. Theorem 4 corresponds to the particular case where $\alpha = 1$, while Diamond (1967)'s preference pattern is more generally compatible with any $\alpha > 0.5$.

7 Pareto indifference

In this section we generalize the results obtained so far by weakening the Pareto principle as follows.

Axiom (Pareto Indifference)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$, if $u_i(x) = u_i(y)$ for all $u_i \in U_i$ and all $i \in I$ then $x \sim_{(U_i)_{i\in I}} y$.

Pareto Indifference only requires the social preference relation to preserve all unanimous individual indifferences. Although the standard Pareto principle may seem mild enough, Pareto Indifference has traditionally been of interest in the social choice literature for at least two reasons. First, it was shown by Harsanyi (1955) to be necessary and sufficient for a linear aggregation of individual utilities. Second, its conjunction with IIA is equivalent to a property known in the SWFL literature as *neutrality* and underlies the proofs of most results obtained in this setting, including Mongin's. The results presented in this section are mathematically the main results of the paper – with the above results under Pareto Preference then following as relatively straightforward corollaries – and require new proof methods that we sketch at the end of the section.

In Mongin and Harsanyi's results, the weakening to Pareto Indifference has the simple effect of allowing individual weights to be negative. In the present ESWFL setting, on the other hand, Theorem 1 holds unchanged under this weakening, so that a utilitarian aggregation – even with negative weights – remains impossible. The impossibility holds, more generally, if IIA and Pareto Indifference are amended as follows.

Axiom (Restricted IIA)**.** For all $i \in I$, all $U_i, V_i \in \mathcal{P}$, and all $x, y \in X$ such that $U_i|_{\{x,y\}} = V_i|_{\{x,y\}}$, $x \succsim_{(U_i,(\{0\})_{j\in I\setminus\{i\}})} y$ if and only if $x \succsim_{(V_i,(\{0\})_{j\in I\setminus\{i\}})} y$.

Axiom (Extended Pareto Indifference)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$, all non-empty $J \subset I$, and all $x, y \in X$, if $x \sim_{((U_i)_{i\in J},(\{0\})_{i\in I\setminus J})} y$ and $x \sim_{((\{0\})_{i\in J},(U_i)_{i\in I\setminus J})} y$ then $x \sim_{(U_i)_{i\in I}} y$.

Restricted IIA weakens IIA by only constraining social preferences when all but one individual have trivial utility sets.[9] Extended Pareto Indifference requires social preferences to preserve all unanimous group indifferences, identifying a group preference with the corresponding social preference when all individuals outside the group have trivial utility sets. Extended Pareto Indifference is stronger than Pareto Indifference under Restricted IIA, but the two are equivalent under IIA and Independence. [10]

Theorem 7. There exists no ESWFL on X satisfying Pareto Indifference, IIA, Completeness, Independence, and Non-Triviality. More generally, there exists no ESWFL on X satisfying Extended Pareto Indifference, Restricted IIA, Completeness, Independence, and Non-Triviality.

[9] Relative utilitarianism (Dhillon, 1998; Dhillon and Mertens, 1999) satisfies Restricted IIA but not IIA.

[10] Indeed, if $u_i(x) = u_i(y)$ for all $u_i \in U_i$ and all $i \in I$, then Restricted IIA yields $x \sim_{(U_i,(\{0\})_{j\in I\setminus\{i\}})} y$ for all $i \in I$ and, hence, repeated application of Extended Pareto Indifference yields $x \sim_{(U_i)_{i\in I}} y$. Conversely, Lemma 6 in the Appendix shows that Pareto Indifference, IIA, and Independence together imply Extended Pareto Indifference.

To generalize Theorem 2, let $\Delta_{2I} = \{(\beta, \gamma) \in \mathbb{R}^{2I}_{+} : \sum_{i \in I} \beta_i + \gamma_i = 1\}$ denote the unit simplex of $\mathbb{R}^{2I}$. Given a subset Φ of Δ_{2I}, let $\langle \Phi \rangle = \mathrm{cl}(\{\mu(\beta, \gamma) - (\kappa, \kappa) \in \Delta_{2I} : \mu \in \mathbb{R}_{+}, \kappa \in \mathbb{R}^{I}_{+}\})$. That is, $\langle \Phi \rangle$ is the set of all limits of sequences of weight vectors in Δ_{2I} that can be obtained from some weight vector $(\beta, \gamma) \in \Phi$ by scaling up all weights by a common factor μ and, for each individual $i \in I$, shifting down both β_i and γ_i by a constant κ_i.[11] Note that $\langle \Phi \rangle$ is compact and convex and that $\Phi \subseteq \langle \Phi \rangle$.

Theorem 8. An ESWFL F^* on X satisfies Pareto Indifference, IIA, Independence, Mixture Continuity, and Non-Triviality if and only if there exists a non-empty, compact, and convex set $\Phi^* \subseteq \Delta_{2I}$ such that for all $(U_i)_{i \in I} \in \mathcal{P}^I$, the vNM utility set

$$U_{\Phi^*, (U_i)_{i \in I}} = \left\{ \sum_{i \in I} \beta_i u_i - \gamma_i v_i : (\beta, \gamma) \in \Phi^*, (u_i, v_i)_{i \in I} \in \prod_{i \in I} U_i^2 \right\}$$

represents $\succsim_{(U_i)_{i \in I}}$. Moreover, another set $\Phi \subseteq \Delta_{2I}$ represents F as above if and only if $\langle \Phi \rangle = \langle \Phi^* \rangle$.

Thus, unlike in Harsanyi's and Mongin's results, the weight vectors under Pareto Indifference feature a positive weight β_i and a negative weight $-\gamma_i$ for each individual $i \in I$, rather than a single positive or negative weight. The uniqueness result asserts that Φ^* is unique up to "redundant" weight vectors, with $\langle \Phi^* \rangle \setminus \Phi^*$ being the set of all weight vectors that are redundant when added to Φ^*. To illustrate this, assume $I = \{1, 2\}$ and let F^* and F be the ESWFs on X represented by $\Phi^* = \{(\beta, \gamma)\}$ and $\Phi = \mathrm{conv}(\{(\beta, \gamma), (\beta', \gamma')\})$, respectively, where

$$\begin{array}{llll} \beta_1 = 0.6, & \gamma_1 = 0.4, & \beta_2 = 0, & \gamma_2 = 0, \\ \beta'_1 = 1, & \gamma'_1 = 0, & \beta'_2 = 0, & \gamma'_2 = 0. \end{array}$$

Note that $\Phi^* \subseteq \Phi = \langle \Phi^* \rangle = \langle \Phi \rangle$. Since $\Phi^* \subseteq \Phi$, $x \succsim_{(U_i)_{i \in I}} y$ implies $x \succsim^*_{(U_i)_{i \in I}} y$ for all $(U_i)_{i \in I} \in \mathcal{P}^I$ and all $x, y \in X$. Conversely, if $x \succsim^*_{(U_i)_{i \in I}} y$ then $0.6u_1(x) - 0.4u_1(x) \geq 0.6u_1(y) - 0.4u_1(y)$, i.e. $u_1(x) \geq u_1(y)$ for all $u_1 \in U_1$ and, hence, $x \succsim_{(U_i)_{i \in I}} y$. So $F = F^*$ as asserted.

To generalize Theorem 3, we strengthen Non-Triviality as follows

Axiom (Egalitarian Non-Triviality)**.** There exist $(U_i)_{i \in I} \in \mathcal{P}^I$ and $x, y \in \hat{X}_{(U_i)_{i \in I}}$ such that $x \nsim_{(U_i)_{i \in I}} y$.

We also let $\hat{\Delta}_{2I} = \{(\beta, \gamma) \in \Delta_{2I} : \sum_{i \in I} \beta_i - \gamma_i \neq 0\}$. Note that for a convex subset Φ of $\hat{\Delta}_{2I}$ – and, hence, for $\langle \Phi \rangle$ as well – we have either $\sum_{i \in I} \beta_i - \gamma_i > 0$ for all $(\beta, \gamma) \in \Phi$ or $\sum_{i \in I} \beta_i - \gamma_i < 0$ for all $(\beta, \gamma) \in \Phi$.

Theorem 9. An ESWFL $F^\wedge$ on X satisfies Pareto Indifference, IIA, Completeness, Egalitarian Independence, Inequality Aversion, Mixture Continuity, and Egalitarian Non-Triviality if and only if there exists a non-empty, compact, and convex set $\Phi^\wedge \subseteq \hat{\Delta}_{2I}$ such that for all $(U_i)_{i \in I} \in \mathcal{P}^I$, the utility function

$$u_{\Phi^\wedge, (U_i)_{i \in I}} : x \mapsto \min_{(\beta, \gamma) \in \Phi^\wedge} \frac{\sum_{i \in I} \beta_i \min_{u_i \in U_i} u_i(x) - \gamma_i \max_{v_i \in U_i} v_i(x)}{\left| \sum_{i \in I} \beta_i - \gamma_i \right|}$$

represents $\succsim^\wedge_{(U_i)_{i \in I}}$. Moreover, another set $\Phi \subseteq \hat{\Delta}_{2I}$ represents F as above if and only if $\langle \Phi \rangle = \langle \Phi^\wedge \rangle$.

To generalize Theorem 5, say that a functional $h : D = \{(s, t) \in \mathbb{R}^{2I} : s \leq t\} \to \mathbb{R}$ is:

[11] Note that we must have $\sum_{i \in I} \mu\beta_i - \kappa_i + \mu\gamma_i - \kappa_i = \mu - 2\sum_{i \in I} \kappa_i = 1$ and, hence, $\mu \geq 1$.

- *weakly constant additive* if $h((s,t)+c) = h(s,t) + h(c)$ for all $(s,t) \in D$ and all $c \in \mathbb{R}$,

- *weakly normalized* if $|h(1)| = 1$,

- *weakly constant linear* if it is positively homogeneous, weakly normalized, and weakly constant additive.

Note that if h is weakly constant linear and monotonic then it is constant linear.

Theorem 10. An ESWFL $F^\wedge$ on X satisfies Pareto Indifference, IIA, Completeness, Egalitarian Independence, Mixture Continuity, and Egalitarian Non-Triviality if and only if there exists a weakly constant linear functional $h : D \to \mathbb{R}$ such that for all $(U_i)_{i\in I} \in \mathcal{P}^I$, the utility function

$$u_{h,(U_i)_{i\in I}} : x \mapsto h\left(\left(\min_{u_i \in U_i} u_i(x)\right)_{i\in I}, \left(\max_{u_i \in U_i} u_i(x)\right)_{i\in I}\right)$$

represents $\succsim^\wedge_{(U_i)_{i\in I}}$. Moreover, h is unique.

Finally, Theorems 4 and 6 generalize straightforwardly. The only additional ingredient needed is a strengthening of Non-Triviality ensuring that $\Phi^* \subset \hat{\Delta}_{2I}$ in Theorem 8. We omit the formal statement of these results.

We now briefly sketch the proofs of this section's results. As in the standard SWFL setting, the conjunction of Pareto Indifference and IIA is equivalent to a neutrality property in our ESWFL setting – see Lemma 1 in the Appendix. In the standard setting, neutrality is further equivalent to the SWFL being *welfarist* in the sense of boiling down to a social ordering over vectors of utility levels. Mongin's result follows directly from this fact by showing that this social ordering satisfies the EU axioms and, hence, can be represented by a linear utility function with respect to individual utililty levels.

In the ESWFL setting, however, neutrality is no longer equivalent to welfarism: the social ranking between two alternatives x and y is fully determined by the restrictions $(U_i|_{\{x,y\}})_{i\in I}$ of individual utility sets to $\{x,y\}$ but not by $(U_i|_{\{x\}})_{i\in I}$ and $(U_i|_{\{y\}})_{i\in I}$ separately. Being therefore unable to extend Mongin's proof method, or other similar arguments from the SWFL literature, we instead rely on two key observations. To state them, given two alternatives $x, y \in X$ and a utility set $U \in \mathcal{P}$, let $U|^x_y = \{u(x) - u(y) : u \in U\} \subset \mathbb{R}$ denote the set of utility differences between and x and y – note that $U|^x_y$ is a compact interval and that $U|^y_x = -U|^x_y$. The first observation is that, under Independence, the social ranking between x and y only depends on the sets $(U_i|^x_y)_{i\in I}$ of individual utility differences between x and y – see Lemma 2 in the Appendix. The second one is that, still under Independence, if $U'_i|^x_y \subseteq U_i|^x_y$ for all $i \in I$ then $x \succsim_{(U_i)_{i\in I}} y$ implies $x \succsim_{(U'_i)_{i\in I}} y$ – with a minor proviso in the absence of Mixture Continuity; see Lemma 3 in the Appendix.

The first claim of Theorem 7 is a direct consequence of these two observations. Indeed, given any profile $(U_i)_{i\in I} \in \mathcal{P}^I$ and alternatives $x, y \in X$, we can find a profile $(V_i)_{i\in I} \in \mathcal{P}^I$ such that $U_i|^x_y \subseteq V_i|^x_y = V_i|^y_x$ for all $i \in I$. By the first observation, then, we can have neither $x \succ_{(V_i)_{i\in I}} y$ nor $y \succ_{(V_i)_{i\in I}} x$, so that we must have $x \sim_{(V_i)_{i\in I}} y$ by Completeness. By the second observation, it follows that $x \sim_{(U_i)_{i\in I}} y$, so that F violates Non-Triviality. To establish the second claim of Theorem 7, we deduce from the first claim and Restricted IIA that F violates Non-Triviality on the subdomain of profiles where all bu one individuals have trivial utility sets and then use Extended Pareto Indifference to extend this to the full domain.

To establish the representation of Theorem 8, using the first observation, we consider the set K of profiles $(U_i|_y^x)_{i\in I}$ such that $x \succsim_{(U_i)_{i\in I}} y$. Each set $U_i|_y^x \subset \mathbb{R}$, being a compact interval, can equivalently be described by the couple $(\min U_i|_y^x, \max U_i|_y^x) \in \mathbb{R}^2$, making K a subset of the finite-dimensional real vector space $\mathbb{R}^{2I}$. By Independence, K is a convex cone and, by Mixture Continuity, K is closed. The set Θ^* of weight vectors representing F^* is then obtained from the polar cone of K in $\mathbb{R}^{2I}$. Finally, the second observation together with Non-Triviality ensures that Θ^* can be taken to be a subset of Δ_{2I}.

To prove Theorems 9 and 10, we cannot rely on the two observations above since $F^\wedge$ does not satisfy Independence. However, by Egalitarian Independence, the first observation still holds when y is egalitarian. We therefore consider the set $\hat{K}$ of profiles $(U_i|_y^x)_{i\in I}$ such that y is egalitarian and $x \succsim_{(U_i)_{i\in I}} y$. $\hat{K}$ is a cone in $\mathbb{R}^{2I}$ by Egalitarian Independence but is not neccessarily convex. However, by Completeness and Egalitarian Non-Triviality, every profile $(U_i|_y^x)_{i\in I}$ admits an "egalitarian equivalent", from which the functional h representing $F^\wedge$ in Theorem 10 is obtained. Finally, Inequality Aversion implies that $\hat{K}$ is convex and that the second observation above still holds when y is egalitarian. The set $\Theta^\wedge$ of weight vectors representing $F^\wedge$ in Theorem 9 is then obtained from the polar cone of $\hat{K}$ in $\mathbb{R}^{2I}$, and the second observation together with Egalitarian Non-Triviality ensure that $\Theta^\wedge$ can be taken to be a subset of $\hat{\Delta}_{2I}$.

8 Special cases

We now analyze three special cases of the above representation theorems, corresponding to particular restrictions on the set of weight vectors. Throughout this section, statements referring to a generic ESWFL F apply indistinctly to coherent or decisive ESWFL. First, we consider the standard Anonymity axiom, which characterizes Bentham (1781)'s classical utilitarianism – all individual having equal weight – in Mongin's theorem. The following notation is needed. A *permutation* of I is a bijection $\pi : I \to I$. Given a vector $s \in \mathbb{R}^I$, we let $\pi(s)$ denote the corresponding permuted vector, i.e. $\pi(s)_i = s_{\pi(i)}$. Similarly, given a profile $(S_i)_{i\in I}$ of sets, we let $\pi((S_i)_{i\in I})$ denote the corresponding permuted profile.

Axiom (Anonymity)**.** For all $(U_i)_{i\in I} \in \mathcal{P}^I$, all $x, y \in X$, and all permutation π of I, $x \succsim_{(U_i)_{i\in I}} y$ if and only if $x \succsim_{\pi((U_i)_{i\in I})} y$.

Proposition 1. In Theorems 8 and 9, F satisfies Anonymity if and only if $(\pi(\beta), \pi(\gamma)) \in \langle\Phi\rangle$ for all $(\beta, \gamma) \in \langle\Phi\rangle$. Hence in Theorems 2 and 3, F satisfies Anonymity if and only if $\pi(\theta) \in \Theta$ for all $\theta \in \Theta$.

Anonymity thus corresponds to the set of weight vectors being closed under permutations. If this set is a singleton – as in Mongin's theorem – then this is equivalent to all individuals having equal weight – or under Pareto Indifference, all individuals having the same positive (resp. negative) weight. More generally, since Φ is convex, it follows that $((\sum_{j\in I} \beta_j/|I|)_{i\in I}, (\sum_{j\in I} \gamma_j/|I|)_{i\in I}) \in \langle\Phi\rangle$ for all $(\beta, \gamma) \in \langle\Phi\rangle$, so Φ contains vectors where all individuals have the same positive (resp. negative) weight. Similarly, since Θ is convex, it follows that $(1/|I|)_{i\in I} \in \Theta$, so Θ contains the equal-weight vector.

Second, we consider the case where all individuals have non-null weights. An individual $i \in I$ is *non-null* if there exist $(U_j)_{j\in I} \in \mathcal{P}^I$ and $x, y \in X$ such that $x \nsim_{(U_j)_{j\in I}} y$ and $u_j(x) = u_j(y)$ for all $j \in I \setminus \{i\}$ and all $u_j \in U_j$. Note that if i is non-null then no individual $j \in I \setminus \{i\}$ is a dictator in the sense of systematically imposing her weak preferences upon society.

Axiom (Full Support)**.** Each individual $i \in I$ is non-null.

Proposition 2. In Theorems 8 and 9, F satisfies Full Support if and only if $\beta + \gamma \gg 0$ for some $(\beta, \gamma) \in \Phi$.[12] Hence in Theorems 2 and 3, F satisfies Full Support if and only if $\theta \gg 0$ for some $\theta \in \Theta$.

Under Pareto Preference, it is common to obtain non-null weights by adding a strict preference clause to the Pareto principle. The following axiom, in particular, ensures that $\theta \gg 0$ in Mongin's theorem.

Axiom (Singleton Pareto Strict Preference)**.** For all $(u_i)_{i\in I} \in P^I$ and all $x, y \in X$, if $u_i(x) \geq u_i(y)$ for all $i \in I$ and $u_i(x) > u_i(y)$ for some $i \in I$ then $x \succ_{(\{u\}_i)_{i\in I}} y$.

Proposition 3. In Theorem 2, F^* satisfies Singleton Pareto Strict Preference if and only if $\theta \gg 0$ for some $\theta \in \Theta^*$. In Theorem 3, $F^\wedge$ satisfies Singleton Pareto Strict Preference if and only if $\theta \gg 0$ for all $\theta \in \Theta^\wedge$.

The case where $\theta \gg 0$ for all $\theta \in \Theta^*$ in Theorem 2 can be characterized by strenghening Singleton Pareto Strict Preference as follows: if $u_i(x) \geq u_i(y)$ for all $i \in I$ and $u_i(x) > u_i(y)$ for some $i \in I$ then for all $z \in X$, there exists $\lambda \in (0,1)$ such that $\lambda x + (1-\lambda)z \succ_{(\{u\}_i)_{i\in I}} y$. The case where $\theta \gg 0$ for some $\theta \in \Theta^\wedge$ in Theorem 3 can be characterized by weakening Singleton Pareto Strict Preference as follows: if $u_j(x) \geq u_j(y) \geq u_i(x) > u_i(y)$ for some $i \in I$ and all $j \in I \setminus \{i\}$ then $x \succ_{(\{u\}_i)_{i\in I}} y$.

Third, we consider the case where the set of weight vectors is a singleton – bringing the ESWFL as close to utilitarianism as possible.

Axiom (Singleton Completeness)**.** For all $(u_i)_{i\in I} \in P^I$, $\succsim_{(\{u_i\})_{i\in I}}$ is complete.

Axiom (Singleton Independence)**.** For all $(u_i)_{i\in I} \in P^I$, $\succsim_{(\{u_i\})_{i\in I}}$ satisfies Independence.

Proposition 4. In Theorem 2, F^* satisfies Singleton Completeness if and only if Θ^* is a singleton. In Theorem 3, $F^\wedge$ satisfies Singleton Independence if and only if $\Theta^\wedge$ is a singleton.

In Theorem 8 (resp. 9), the special case where Φ is a singleton implies but is not implied by Singleton Completeness (resp. Singleton Independence). To see that it is not implied, assume $I = \{1,2\}$ and let F be the ESWFL represented as per Theorem 8 (resp. 9) by $\Phi = \mathrm{conv}(\{(\beta,\gamma),(\beta',\gamma')\})$, where

$$\begin{aligned} \beta_1 &= 0.4, & \gamma_1 &= 0.3, & \beta_2 &= 0.2, & \gamma_2 &= 0.1, \\ \beta'_1 &= 0.2, & \gamma'_1 &= 0.1, & \beta'_2 &= 0.4, & \gamma'_2 &= 0.3. \end{aligned}$$

Then $x \succsim_{(u_i)_{i\in I}} y$ if and only if $u_1(x) + u_2(x) \geq u_1(y) + u_2,(y)$ for all $(u_i)_{i\in I} \in P^I$ and all $x, y \in X$, so F satisfies Singleton Completeness (resp. Singleton Independence). However, there exists no $(\beta'', \gamma'') \in \Delta_{2I}$ such that $\langle\{(\beta'',\gamma'')\}\rangle = \langle\Phi\rangle$.[13]

9 Interpersonal utility comparisons

Most of the SWFL literature deals with abstract – or riskless – alternatives and imposes various "informational invariance" axioms (see e.g. D'Aspremont and Gevers, 1977; Maskin, 1978; Roberts, 1980; Blackorby et al., 1984). These axioms express the degree of measurability and interpersonal comparability of utility by limiting the responsiveness of social preferences to transformations of the individual

[12]Note that $\beta + \gamma \gg 0$ for some $(\beta,\gamma) \in \Phi$ if and only if $\beta + \gamma \gg 0$ for some $(\beta,\gamma) \in \langle\Phi\rangle$.

[13]Indeed, $(\beta'',\gamma'') \in \langle\Phi\rangle$ implies $\beta''_1 - \gamma''_1 = \beta''_2 - \gamma''_2 \geq 0.1$ whereas $\Phi \subseteq \langle\{(\beta'',\gamma'')\}\rangle$ implies $\beta''_1 - \gamma''_1 = \beta''_2 - \gamma''_2 < 0.1$.

utility profile. In particular, representation theorems for utilitarian SWFLs typically require a "Cardinal Measurability / Unit Comparability" axiom, whereas representation theorems for egalitarian SWFLs typically require an "Ordinal Measurability / Full Comparability") axiom. As is the case for Mongin's theorem, the former turns out to be redundant and, more precisely, equivalent to Independence in Theorems 2 and 8. Moreover, the same is true for the latter and Egalitarian Independence in Theorems 3 and 9, with the caveat that utility is cardinally rather than ordinally measurable. Before stating these results, we generalize these two axioms in our extended setting.

Axiom (Cardinal Measurability / Unit Comparability – CU)**.** For all $(U_i)_{i\in I}, (V_i)_{i\in I} \in \mathcal{P}^I$, if there exist $a \in \mathbb{R}_{++}$ and $(b_i : U_i \to \mathbb{R})_{i\in I}$ such that $V_i = \{au_i + b_i(u_i) : u_i \in U_i\}$ for all $i \in I$ then $\succsim_{(U_i)_{i\in I}} = \succsim_{(V_i)_{i\in I}}$.

Axiom (Cardinal Measurability / Full Comparability – CF)**.** For all $(U_i)_{i\in I}, (V_i)_{i\in I} \in \mathcal{P}^I$, if there exist $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $V_i = \{au_i + b : u_i \in U_i\}$ for all $i \in I$ then $\succsim_{(U_i)_{i\in I}} = \succsim_{(V_i)_{i\in I}}$.

Proposition 5. Let F be an ESWFL satisfying Pareto Indifference and IIA. Then F satisfies CU (resp. CF) if and only if it satisfies Independence (resp. Egalitarian Independence).

In the specific context of risky alternatives, Theorems 2 and 8 also shed light on the type and degree of interpersonal comparability of utility implicitly assumed in Harsanyi's Aggregation Theorem. Whereas Harsanyi (1979, p294) considered that his theorem does not rely on such assumptions, Broome (1991, p219–220) argued that the possibility of such comparison is implicit in the assumption that social preferences be complete. Mongin (1994, p350), on the other hand, suggested that this possibility might as well be embodied in the restriction to profile of single utility functions rather classes of utility functions. Theorems 2 and 8 show that a partial form of utilitarianism remains when social preferences are incomplete and profiles of utility sets are considered.

10 Conclusion

The present paper extends Mongin (1994)'s multi-profile version of Harsanyi (1955)'s Aggregation Theorem by allowing individual preferences to be incomplete. An impossibility result was first established, implying that social preferences cannot be utilitarian in this extended setting. Two forms of partial utilitarianism were then characterized by relaxing the expected utility axioms at the social level: a coherent one relying on unanimity across a set of utilitarian criteria and a decisive one relying on the least favorable of these criteria. A more general form of decisive partial utilitarianism, in the spirit of Hurwicz (1951), was also characterized.

Distinguishing between a social planner's coherent and decisive preferences allows in a sense to retain both completeness and independence, albeit not simultaneously. An alternative resolution of the tension between these two axioms could consist in a single social preference relation satisfying neither of them and reflecting some compromise between coherence and decisiveness. This social preference relation could for instance be represented by a collection Ω of sets of weight vectors, in the sense that $x \succsim_{(U_i)_{i\in I}} y$ if and only if $\min_{\theta\in\Theta} \sum_{i\in I} \theta_i \min_{u_i\in U_i} u_i(x) \geq \min_{\theta\in\Theta} \sum_{i\in I} \theta_i \min_{u_i\in U_i} u_i(y)$ for all $\Theta \in \Omega$. A similar representation was characterized by Nascimento and Riella (2011) in the context of individual decision making under uncertainty. Whether the present setting is rich enough to allow for such a characterization is an open question.

A Appendix: proofs

We first prove the most general results under Pareto Indifference (Theorems 7–10 in Section 7), then obtain the results under Pareto Preference (Theorems 1–6 in Sections 4–6) as corollaries and, finally, establish Propositions 1–5 in Sections 8 and 9. All proofs are stated for a generic ESWFL F, except for Theorems 4 and 6 where two ESWFLs F^* and $F^\wedge$ need to be considered simultaneously.

A.1 Proof of Theorem 7

Although it would suffice to prove the second claim directly, we proceed by proving the first one – as several intermediate lemmas will be useful later on – and then using it to establish the second one. So assume that F satisfies Pareto Indifference, IIA, Completeness, and Independence. We will show that F violates Non-Triviality. We start by showing that the social ranking between two alternatives only depends on the restriction of individual utility sets to these alternatives.

Lemma 1. For all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y, x', y' \in X$ such that $U_i|_{\{x,y\}} = U'_i|_{\{x',y'\}}$ for all $i \in I$, $x \succsim_{(U_i)_{i\in I}} y$ if and only if $x' \succsim_{(U'_i)_{i\in I}} y'$.

Proof. We first claim that the result holds when $y = y'$. Since the affine dimension of X is at least 2, there exists $z \in X$ such that both (x, y, z) and (x', y, z) are affinely independent. Let Y and Y' be two affine bases of X containing $\{x, y, z\}$ and $\{x', y, z\}$, respectively. For all $u \in P$, define $v_u, v'_u \in P$ by

$$v_u(x) = u(x), \qquad v_u(y) = u(y), \qquad v_u(z) = u(x), \qquad v_u(w) = 0 \text{ for all } w \in Y \setminus \{x, y, z\},$$
$$v'_u(x') = u(x'), \qquad v'_u(y) = u(y), \qquad v'_u(z) = u(x'), \qquad v'_u(w) = 0 \text{ for all } w \in Y' \setminus \{x', y, z\}.$$

For all $i \in I$, let $V_i = \{v_u : u \in U_i\}$ and $V'_i = \{v'_u : u \in U_i\}$. Then $V_i, V'_i \in \mathcal{P}$ and

$$U_i|_{\{x,y\}} = V_i|_{\{x,y\}} = V_i|_{\{z,y\}} = V'_i|_{\{z,y\}} = V'_i|_{\{x',y\}} = U'_i|_{\{x',y\}}.$$

Moreover, $x \sim_{(V_i)_{i\in I}} z$ and $x' \sim_{(V'_i)_{i\in I}} z$ by Pareto indifference. Hence

$$x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow x \succsim_{(V_i)_{i\in I}} y \Leftrightarrow z \succsim_{(V_i)_{i\in I}} y \Leftrightarrow z \succsim_{(V'_i)_{i\in I}} y \Leftrightarrow x' \succsim_{(V'_i)_{i\in I}} y \Leftrightarrow x' \succsim_{(U'_i)_{i\in I}} y,$$

where the first, third, and fifth equivalences follow from IIA and the second and fourth ones from transitivity of $\succsim_{(V_i)_{i\in I}}$ and $\succsim_{(V'_i)_{i\in I}}$, respectively. Similarly,

$$y \succsim_{(U_i)_{i\in I}} x \Leftrightarrow y \succsim_{(V_i)_{i\in I}} x \Leftrightarrow y \succsim_{(V_i)_{i\in I}} z \Leftrightarrow y \succsim_{(V'_i)_{i\in I}} z \Leftrightarrow y \succsim_{(V'_i)_{i\in I}} x' \Leftrightarrow y \succsim_{(U'_i)_{i\in I}} x',$$

which proves the claim.

Now assume $y \neq y'$. Let Y be an affine basis of X containing $\{x', y\}$. For all $u \in P$, define $v_u \in P$ by

$$v_u(x') = u(x), \qquad v_u(y) = u(y), \qquad v_u(z) = 0 \text{ for all } z \in Y \setminus \{x', y\}.$$

For all $i \in I$, let $V_i = \{v_u : u \in U_i\}$. Then $V_i \in \mathcal{P}$ and $U_i|_{\{x,y\}} = V_i|_{\{x',y\}} = U'_i|_{\{x',y'\}}$. Hence

$$x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow x' \succsim_{(V_i)_{i\in I}} y \Leftrightarrow x' \succsim_{(U'_i)_{i\in I}} y'$$

by the above claim, which completes the proof. □

We now further show that the social ranking between two alternatives only depends on the individual sets of utility differences between these two alternatives.

Lemma 2. For all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y, x', y' \in X$ such that $U_i|^x_y = U'_i|^{x'}_{y'}$ for all $i \in I$, $x \succsim_{(U_i)_{i\in I}} y$ if and only if $x' \succsim_{(U'_i)_{i\in I}} y'$.

Proof. Since the affine dimension of X is at least 2, there exists $z \in X$ such that both (x, y, z) and (x', y', z) are affinely independent. Let Y and Y' be two affine bases of X containing $\{x, y, z\}$ and $\{x', y', z\}$, respectively. For all $u \in P$, define $v_u, v'_u \in P$ by

$$\begin{aligned} &v_u(x) = u(x), && v_u(y) = u(y), && v_u(z) = -u(y), && v_u(w) = 0 \text{ for all } w \in Y \setminus \{x, y, z\}, \\ &v'_u(x') = u(x'), && v'_u(y') = u(y'), && v'_u(z) = -u(y'), && v'_u(w) = 0 \text{ for all } w \in Y' \setminus \{x', y', z\}. \end{aligned}$$

For all $i \in I$, let $V_i = \{v_u : u \in U_i\}$ and $V'_i = \{v'_u : u \in U'_i\}$. Then $V_i, V'_i \in \mathcal{P}$, $V_i|_{\{x,y\}} = U_i|_{\{x,y\}}$, $V'_i|_{\{x',y'\}} = U'_i|_{\{x',y'\}}$, and

$$V_i|_{\{0.5x+0.5z, 0.5y+0.5z\}} = 0.5U_i|^x_y \times \{0\} = 0.5U'_i|^{x'}_{y'} \times \{0\} = V'_i|_{\{0.5x'+0.5z, 0.5y'+0.5z\}}.$$

Hence

$$\begin{aligned} x \succsim_{(U_i)_{i\in I}} y &\Leftrightarrow x \succsim_{(V_i)_{i\in I}} y \\ &\Leftrightarrow 0.5x + 0.5z \succsim_{(V_i)_{i\in I}} 0.5y + 0.5z \\ &\Leftrightarrow 0.5x' + 0.5z \succsim_{(V'_i)_{i\in I}} 0.5y' + 0.5z \\ &\Leftrightarrow x' \succsim_{(V'_i)_{i\in I}} y' \\ &\Leftrightarrow x' \succsim_{(U'_i)_{i\in I}} y', \end{aligned}$$

where the first and fifth equivalences follow from IIA, the second and fourth ones from the fact that $\succsim_{(V_i)_{i\in I}}$ and $\succsim_{(V'_i)_{i\in I}}$ satisfy Independence, and the third one from Lemma 1. □

Next, we show that a social weak preference persists when the individual sets of utility differences shrink in a "non-degenerate" way.

Lemma 3. For all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$ such that, for all $i \in I$, $\lambda_i \min U_i|^x_y + (1 - \lambda_i) \max U_i|^x_y \in U'_i|^x_y \subseteq U_i|^x_y$ for some $\lambda_i \in (0, 1)$, if $x \succsim_{(U_i)_{i\in I}} y$ then $x \succsim_{(U'_i)_{i\in I}} y$.

Proof. We first claim that the result holds in the particular case where $\lambda_i = 0.5$ for all $i \in I$. Since the affine dimension of X is at least 2, there exists $z \in X$ such that (x, y, z) are affinely independent. Let Y be an affine basis of X containing $\{x, y, z\}$. For all $i \in I$, define $u_i, u'_i, v_i, v'_i \in P$ by

$$u_i(x) = \min U_i|^x_y, \qquad u_i(y) = 2 \min U'_i|^x_y - \min U_i|^x_y, \qquad u_i(w) = 0 \text{ for all } w \in Y \setminus \{x, y\},$$

$$
\begin{aligned}
&u_i'(x) = 2\min U_i'|_y^x - \min U_i|_y^x, && u_i'(y) = \min U_i|_y^x, && u_i'(w) = 0 \text{ for all } w \in Y \setminus \{x, y\},\\
&v_i(x) = \max U_i|_y^x, && v_i(y) = 2\max U_i'|_y^x - \max U_i|_y^x, && v_i(w) = 0 \text{ for all } w \in Y \setminus \{x, y\},\\
&v_i'(x) = 2\max U_i'|_y^x - \max U_i|_y^x, && v_i'(y) = \max U_i|_y^x, && v_i'(w) = 0 \text{ for all } w \in Y \setminus \{x, y\}.
\end{aligned}
$$

Let $V_i = \mathrm{conv}(\{u_i, u_i', v_i, v_i'\})$. Then $V_i \in \mathcal{P}$, $V_i|_z^x = V_i|_z^y = U_i|_y^x$, and $V_i|_z^{0.5x+0.5y} = U_i'|_y^x$. Hence

$$
\begin{aligned}
x \succsim_{(U_i)_{i\in I}} y &\Leftrightarrow x \succsim_{(V_i)_{i\in I}} z, y \succsim_{(V_i)_{i\in I}} z\\
&\Rightarrow 0.5x + 0.5y \succsim_{(V_i)_{i\in I}} z\\
&\Leftrightarrow x \succsim_{(U_i')_{i\in I}} y,
\end{aligned}
$$

where the two equivalences follow from Lemma 2 and the implication from the fact that $\succsim_{(V_i)_{i\in I}}$ is transitive and satisfies Independence. This proves the claim.

Now for the general case, for all $i \in I$, let $[s_i, t_i] = U_i|_y^x$ and $[s_i', t_i'] = U_i'|_y^x$. We construct a sequence $([s_i^n, t_i^n])_{n\in\mathbb{N}}$ as follows:

- $[s_i^0, t_i^0] = [s_i', t_i']$,
- If $s_i < t_i$ and $s_i' = t_i'$ then $[s_i', t_i'] \subset [s_i^1, t_i^1] = [s_i' - c_i, t_i' + c_i] \subseteq [s_i, t_i]$ for some $c_i > 0$, otherwise $[s_i^1, t_i^1] = [s_i', t_i']$.
- For all $n \geq 1$, if $2(t_i^n - s_i^n) < t_i - s_i$ then $[s_i^n, t_i^n] \subset [s_i^{n+1}, t_i^{n+1}] \subset [s_i, t_i]$ and $t_i^{n+1} - s_i^{n+1} = 2(t_i^n - s_i^n)$, otherwise $[s_i^{n+1}, t_i^{n+1}] = [s_i, t_i]$.

We then have $0.5s_i^{n+1} + 0.5t_i^{n+1} \in [s_i^n, t_i^n] \subseteq [s_i^{n+1}, t_i^{n+1}]$ for all $n \in \mathbb{N}$. Moreover, since I is finite, there exists $n^* \in \mathbb{N}$ such that $[s_i^n, t_i^n] = [s_i, t_i]$ for all $i \in I$ and all $n \geq n^*$. For all $n \in \mathbb{N}$, define $u_i^n, v_i^n \in P$ by

$$
\begin{aligned}
&u_i^n(x) = s_i^n, && u_i^n(w) = 0 \text{ for all } w \in Y \setminus \{x\},\\
&v_i^n(x) = t_i^n, && v_i^n(w) = 0 \text{ for all } w \in Y \setminus \{x\}.
\end{aligned}
$$

Let $U_i^n = \mathrm{conv}(\{u_i^n, v_i^n\})$. Then $U_i^n \in \mathcal{P}$ and $U_i^n|_y^x = [s_i^n, t_i^n]$. The result then follows from repeated application of the above claim. □

The next lemma shows that F violates Non-Triviality, which completes the proof of the first claim.

Lemma 4. For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$, $x \sim_{(U_i)_{i\in I}} y$.

Proof. Let Y be an affine basis of X containing $\{x, y\}$. For all $t \in \mathbb{R}$, define $v_t \in P$ by

$$
v_t(x) = t, \qquad v_t(y) = 0 \text{ for all } y \in Y \setminus \{x\}.
$$

For all $i \in I$, let $s_i \in \mathbb{R}_+$ be such that $U_i|_y^x \subset (-s_i, s_i)$ and let $V_i = \{v_t : t \in [-s_i, s_i]\}$. Then $V_i \in \mathcal{P}$ and $V_i|_y^x = V_i|_x^y = [-s_i, s_i]$. Hence if $y \succ_{(V_i)_{i\in I}} x$ then $x \succ_{(V_i)_{i\in I}} y$ by Lemma 2, a contradiction, and vice versa. Since $\succsim_{(V_i)_{i\in I}}$ is complete, it follows that $x \sim_{(V_i)_{i\in I}} y$ and, hence, $x \sim_{(U_i)_{i\in I}} y$ by Lemma 3. □

Finally, to prove the second claim, assume that F satisfies Extended Pareto Indifference, Restricted IIA, Completeness, and Independence. For all $i \in I$, consider a population made of individual i alone and define the ESWFL F_i on X by $F_i(U_i) = F(U_i, (\{0\})_{j\in I\setminus\{i\}})$ for all $U_i \in \mathcal{P}$. Clearly, F_i satisfies Pareto Indifference, IIA, Completeness, and Independence. By the first claim, it follows that $x \sim_{(U_i,(\{0\})_{j\in I\setminus\{i\}})} y$ for all $x, y \in X$ and all $U_i \in \mathcal{P}$. Hence $x \sim_{(U_i)_{i\in I}} y$ for all $x, y \in X$ and all $(U_i)_{i\in I} \in \mathcal{P}^I$ by repeated application of Extended Pareto Indifference, so that F violates Non-Triviality.

A.2 Proof of Theorem 8

Clearly, if there exists a non-empty, compact, and convex set $\Phi \subseteq \Delta_{2I}$ representing F then F satisfies Pareto Indifference, IIA, Independence, Mixture Continuity, and Non-Triviality. Conversely, assume F satisfies these axioms. First note that Lemmas 1, 2, and 3 hold since their proofs do not rely on Completeness. Let

$$
\begin{aligned}
D &= \left\{(s,t) \in \mathbb{R}^{2I} \,:\, s \le t\right\},\\
E &= \left\{((\min U_i|^x_y)_{i\in I}, (\max U_i|^x_y)_{i\in I}) \in \mathbb{R}^{2I} \,:\, (U_i)_{i\in I} \in \mathcal{P}^I, x, y \in X\right\},\\
K &= \left\{((\min U_i|^x_y)_{i\in I}, (\max U_i|^x_y)_{i\in I}) \in \mathbb{R}^{2I} \,:\, (U_i)_{i\in I} \in \mathcal{P}^I, x, y \in X, x \succsim_{(U_i)_{i\in I}} y\right\}.
\end{aligned}
$$

E essentially consists of all profiles $(U_i|^x_y)_{i\in I}$ of individual sets of utility differences corresponding to some profile $(U_i)_{i\in I} \in \mathcal{P}^I$ of individual utility sets and some alternatives $x, y \in X$, whereas K essentially consists of those profiles $(U_i|^x_y)_{i\in I} \in E$ for which $x \succsim_{(U_i)_{i\in I}} y$. It is easy to see that $K \subseteq E$ and that $E = D$ is a non-empty, closed, and convex cone. Moreover, F is fully determined by K in the following sense.

Lemma 5. For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$,

$$x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow ((\min U_i|^x_y)_{i\in I}, (\max U_i|^x_y)_{i\in I}) \in K.$$

Proof. The "if" part holds by definition of K. The "only if" part follows from Lemma 2. □

We now establish some properties of the set K.

Lemma 6. K is a non-empty, closed, and convex cone.[14]

Proof. First, by Pareto Indifference, $0 \in K$, so K is non-empty. Second, we show that K is a cone, i.e. for all $(s,t) \in D$ and all $\lambda \in (0,1)$, $(s,t) \in K$ if and only if $\lambda(s,t) \in K$. Let Y be an affine basis of X and $x, y \in Y$. For all $i \in I$, define $u_i, v_i \in P$ by

$$
\begin{aligned}
u_i(x) &= s_i, \qquad & u_i(z) &= 0 \text{ for all } z \in Y \setminus \{x\},\\
v_i(x) &= t_i, \qquad & v_i(z) &= 0 \text{ for all } z \in Y \setminus \{x\}.
\end{aligned}
$$

Let $U_i = \mathrm{conv}(\{u_i, v_i\})$. Then $U_i \in \mathcal{P}$, $U_i|^x_y = [s_i, t_i]$, and $U_i|^{\lambda x+(1-\lambda)y}_y = \lambda[s_i, t_i]$. Hence

$$(s,t) \in K \Leftrightarrow x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow \lambda x + (1-\lambda)y \succsim_{(U_i)_{i\in I}} y \Leftrightarrow \lambda(s,t) \in K,$$

[14] The fact that K is closed under addition – which does not rely on Mixture Continuity – implies that F satisfies Extended Pareto Indifference.

where the first and third equivalences follow from Lemma 5 and the second one from the fact that $\succsim_{(U_i)_{i\in I}}$ satisfies Independence.

Third, we show that K is convex, i.e. $\lambda(s,t)+(1-\lambda)(s',t')\in K$ for all $(s,t),(s',t')\in K$ and all $\lambda\in(0,1)$. Let Y be an affine basis of X and $x,y,z\in Y$. For all $i\in I$, define $u_i,v_i\in P$ by

$$\begin{aligned}
&u_i(x)=s_i, && u_i(y)=s_i', && u_i(w)=0 \text{ for all } w\in Y\setminus\{x,y\},\\
&v_i(x)=t_i, && v_i(y)=t_i', && v_i(w)=0 \text{ for all } w\in Y\setminus\{x,y\}.
\end{aligned}$$

Let $U_i=\operatorname{conv}(\{u_i,v_i\})$. Then $U_i\in\mathcal{P}$, $U_i|_z^x=[s_i,t_i]$, $U_i|_z^y=[s_i',t_i']$, and $U_i|_z^{\lambda x+(1-\lambda)y}=\lambda[s_i,t_i]+(1-\lambda)[s_i',t_i']$. Hence

$$\begin{aligned}
(s,t),(s',t')\in K &\Leftrightarrow x\succsim_{(U_i)_{i\in I}} z,\, y\succsim_{(U_i)_{i\in I}} z\\
&\Rightarrow \lambda x+(1-\lambda)y\succsim_{(U_i)_{i\in I}} z\\
&\Leftrightarrow \lambda(s,t)+(1-\lambda)(s',t')\in K,
\end{aligned}$$

where the two equivalences follow from Lemma 5 and the implication from the fact that $\succsim_{(U_i)_{i\in I}}$ is transitive and satisfies Independence.

Finally, we show that K is closed (in $\mathbb{R}^{2I}$ or, equivalently, in D), i.e. that it contains its closure. Let $(s,t)\in D$ belong to the closure of K. Since K is non-empty and convex, it has a non-empty relative interior. Let $(s',t')\in D$ belong to the relative interior of K. Then for all $\lambda\in(0,1)$, $\lambda(s,t)+(1-\lambda)(s',t')\in K$ (Rockafellar, 1970, Theorem 6.1). Let $Y,x,y,z,(U_i)_{i\in I}$ be as in the previous paragraph. It follows that for all $\lambda\in(0,1)$, $\lambda x+(1-\lambda)y\succsim_{(U_i)_{i\in I}} z$ by Lemma 5. Hence $x\succsim_{(U_i)_{i\in I}} z$ since $\succsim_{(U_i)_{i\in I}}$ is mixture continuous and, hence, $(s,t)\in K$ by Lemma 5. □

Lemma 7. For all $(s,t),(s',t')\in D$ such that $[s_i',t_i']\subseteq[s_i,t_i]$ for all $i\in I$, if $(s,t)\in K$ then $(s',t')\in K$.

Proof. We first claim that the result holds if for all $i\in I$, $\lambda_i s_i+(1-\lambda_i)t_i\in[s_i',t_i']$ for some $\lambda_i\in(0,1)$. This claim follows directly from Lemma 3.

Now assume that for some $i\in I$, $\lambda_i s_i+(1-\lambda_i)t_i\notin[s_i',t_i']$ for all $\lambda_i\in(0,1)$. Note that this implies that $s_i<t_i$. Then by the above claim, $\lambda(s,t)+(1-\lambda)(s',t')\in K$ for all $\lambda\in(0,1)$. Hence $(s',t')\in K$ since K is closed by Lemma 6. □

Now, given a cone C in $\mathbb{R}^{2I}$, let

$$C^*=\left\{(\beta,\gamma)\in\mathbb{R}^{2I}\,:\,\forall(s,t)\in C,\sum_{i\in I}\beta_i s_i-\gamma_i t_i\ge 0\right\}$$

denote the polar cone of C.[15] Given a subset Φ of Δ_{2I}, let

$$K_\Phi=\left\{(s,t)\in D\,:\,\forall(\beta,\gamma)\in\Phi,\sum_{i\in I}\beta_i s_i-\gamma_i t_i\ge 0\right\}. \tag{1}$$

Then C^* and K_Φ are non-empty, closed, and convex cones.

Lemma 8. $D^*=\{-(\kappa,\kappa)\,:\,\kappa\in\mathbb{R}_+^I\}$ and $K^*=\operatorname{cone}(K^*\cap\Delta_{2I})+D^*$.

[15]More precisely, C^* is the image of the polar cone of C under the transformation $(\beta,\gamma)\mapsto(\beta,-\gamma)$.

Proof. To prove the former equality, first note that for all $\kappa \in \mathbb{R}^I_+$ and all $(s,t) \in E$, $\sum_{i\in I} -\kappa_i(s_i - t_i) \geq 0$ since $s_i \leq t_i$ for all $i \in I$. Conversely, let $(\beta,\gamma) \in \mathbb{R}^{2I} \setminus \{-(\kappa,\kappa) : \kappa \in \mathbb{R}^I_+\}$. Then either $\beta_i \neq \gamma_i$ or $\beta_i = \gamma_i > 0$ for some $i \in I$. In the former case, letting $s_i - t_i = \gamma_i - \beta_i$ and $s_j = t_j = 0$ for all $j \in I \setminus \{i\}$, we have $\sum_{j\in I} \beta_j s_j - \gamma_j t_j = (\beta_i - \gamma_i)(\gamma_i - \beta_i) < 0$, so $(\beta,\gamma) \notin D^*$. In the the latter case, letting $s_1 = -1$, $t_i = 1$, and $s_j = t_j = 0$ for all $j \in I \setminus \{i\}$, we have $\sum_{j\in I} \beta_j s_j - \gamma_j t_j = -\beta_i - \gamma_i < 0$, so again $(\beta,\gamma) \notin D^*$.

For the latter equality, we fist claim that that $K^* = (K^* \cap \mathbb{R}^{2I}_+) + D^*$. To prove this claim, first note that $K^* \cap \mathbb{R}^{2I}_+ \subseteq K^*$ by definition. Moreover, for all $\kappa \in \mathbb{R}^I_+$ and all $(s,t) \in K$, $\sum_{i\in I} -\kappa_i(s_i - t_i) \geq 0$ since $\kappa_i \geq 0$ and $s_i \leq t_i$ for all $i \in I$, so that $D^* \subseteq K^*$. Since K^* is a convex cone, it follows that $(K^* \cap \mathbb{R}^{2I}_+) + D^* \subseteq K^*$. Conversely, let $(\beta,\gamma) \in K^*$. Let

$$J = \left\{i \in I : \beta_i \geq 0, \gamma_i \geq 0\right\}, \quad J' = \left\{i \in I \setminus J : \beta_i \geq \gamma_i\right\}, \quad J'' = \left\{i \in I \setminus J : \beta_i < \gamma_i\right\}.$$

Then (J, J', J'') is a partition of I, $\gamma_i < 0$ for all $i \in J'$, and $\beta_i < 0$ for all $i \in J''$. Define $\beta', \gamma', \kappa \in \mathbb{R}^I$ by

$$\begin{aligned}
&\beta'_i = \beta_i \text{ for all } i \in J, && \beta'_i = \beta_i - \gamma_i \text{ for all } i \in J', && \beta'_i = 0 \text{ for all } i \in J'',\\
&\gamma'_i = \gamma_i \text{ for all } i \in J, && \gamma'_i = 0 \text{ for all } i \in J', && \gamma'_i = \gamma_i - \beta_i \text{ for all } i \in J'',\\
&\kappa_i = 0 \text{ for all } i \in J, && \kappa_i = -\gamma_i \text{ for all } i \in J', && \kappa_i = -\beta_i \text{ for all } i \in J''.
\end{aligned}$$

Then $\beta'_i \geq 0$, $\gamma'_i \geq 0$, and $\kappa_i \geq 0$ for all $i \in I$. Moreover, $(\beta,\gamma) = (\beta',\gamma') - (\kappa,\kappa)$, so it is sufficient to prove that $(\beta',\gamma') \in K^*$. To this end, let $(s,t) \in K$. We need to show that $\sum_{i\in I} \beta'_i s_i - \gamma'_i t_i \geq 0$. Define $(s',t') \in D$ by

$$\begin{aligned}
&s'_i = s_i \text{ for all } i \in J, && s'_i = s_i \text{ for all } i \in J', && s'_i = t_i \text{ for all } i \in J'',\\
&t'_i = t_i \text{ for all } i \in J, && t'_i = s_i \text{ for all } i \in J', && t'_i = t_i \text{ for all } i \in J''.
\end{aligned}$$

Then $[s'_i, t'_i] \subseteq [s_i, t_i]$ for all $i \in I$ and, hence, $(s',t') \in K$ by Lemma 7, so that $\sum_{i\in I} \beta_i s'_i - \gamma_i t'_i \geq 0$ by (1). Moreover,

$$\sum_{i\in I} \beta_i s'_i - \gamma_i t'_i = \sum_{i\in J} \beta_i s_i - \gamma_i t_i + \sum_{i\in J'} (\beta_i - \gamma_i) s_i + \sum_{i\in J''} (\gamma_i - \beta_i) t_i = \sum_{i\in I} \beta'_i s_i - \gamma'_i t_i,$$

so that $\sum_{i\in I} \beta'_i s_i - \gamma'_i t_i \geq 0$, which completes the proof of the claim. Finally, note that if $K^* \cap \mathbb{R}^{2I}_+ = \{0\}$ then $K^* = D^*$ and, hence, $K = D$, contradicting Non-Triviality by Lemma 5. Hence $K^* \cap \mathbb{R}^{2I}_+ \neq \{0\}$, and, hence, $K^* \cap \mathbb{R}^{2I}_+ = \text{cone}(K^* \cap \Delta_{2I})$, which completes the proof. □

Lemma 9. A set $\Phi \subseteq \Delta_{2I}$ represents F if and only if $= \text{cl}(\text{cone}(\Phi) + D^*) = K^*$.

Proof. We have $K^*_\Phi = \text{cl}(\text{cone}(\Phi) + D^*)$ (Rockafellar, 1970, Corollary 16.4.2), so that $K = K_\Phi$ if and only if $K^* = \text{cl}(\text{cone}(\Phi) + D^*)$. Moreover, by Lemma 5, we have $K = K_\Phi$ if and only if for all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$,

$$x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow \left[\forall(\beta,\gamma) \in \Phi, \sum_{i\in I} \beta_i \min U_i|^x_y - \gamma_i \max U_i|^x_y \geq 0\right]$$

$$\Leftrightarrow \left[\forall(\beta,\gamma) \in \Phi, \forall(u_i, v_i)_{i\in I} \in \prod_{i\in I} U_i^2, \sum_{i\in I} \beta_i(u_i(x) - u_i(y)) - \gamma_i(v_i(x) - v_i(y)) \geq 0\right]$$

$$\Leftrightarrow \left[\forall(\beta,\gamma) \in \Phi, \forall(u_i, v_i)_{i\in I} \in \prod_{i\in I} U_i^2, \sum_{i\in I} \beta_i u_i(x) - \gamma_i v_i(x) \geq \sum_{i\in I} \beta_i u_i(y) - \gamma_i v_i(y)\right]$$

$$\Leftrightarrow \left[\forall u \in U_{\Phi,(U_i)_{i\in I}}, u(x) \geq u(y)\right].$$

Hence $K = K_\Phi$ if and only if Φ represents F. □

Let $\Phi = K^* \cap \Delta_{2I}$. Then Φ is compact and convex since K^* is closed and convex and Δ_{2I} is compact and convex. Moreover, $K^* = \mathrm{cone}(\Phi) + D^* = \mathrm{cl}(\mathrm{cone}(\Phi) + D^*)$ by Lemma 8 and since K^* is closed. Since K^* is non-empty, it follows that Φ is non-empty as well. This establishes the main result by Lemma 9, so we turn to the uniqueness claim.

Lemma 10. For all $\Phi \subseteq \Delta_{2I}$, $K_\Phi^* = \mathrm{cone}(\langle\Phi\rangle) + D^*$ and $\langle\Phi\rangle = K_\Phi^* \cap \Delta_{2I}$.

Proof. To prove the former equality, first note that $\langle\Phi\rangle \subset \mathrm{cl}(\mathrm{cone}(\Phi) + D^*) = K_\Phi^*$ by definition and, hence, $\mathrm{cone}(\langle\Phi\rangle) + D^* \subseteq K_\Phi^*$ since K_Φ^* is a convex cone containing D^*. Conversely, first note that $D^* \subseteq \mathrm{cone}(\langle\Phi\rangle) + D^*$ by definition. For all $(\beta,\gamma) \in \mathbb{R}^{2I} \setminus D^*$, define $\kappa_{(\beta,\gamma)} \in \mathbb{R}^I$, $\mu_{(\beta,\gamma)} \in \mathbb{R}$, and $\phi_{(\beta,\gamma)} \in \mathbb{R}^{2I}$ by

$$\kappa_{(\beta,\gamma)} = (\max\{0, -\beta_i, -\gamma_i\})_{i\in I}, \quad \mu_{(\beta,\gamma)} = \sum_{i\in I} \beta_i + \gamma_i + 2\kappa_{(\beta,\gamma)_i}, \quad \phi_{(\beta,\gamma)} = \frac{(\beta,\gamma) + (\kappa_{(\beta,\gamma)}, \kappa_{(\beta,\gamma)})}{\mu_{(\beta,\gamma)}}.$$

Note that $\kappa_{(\beta,\gamma)} \geq 0$ and $(\beta,\gamma) + (\kappa_{(\beta,\gamma)}, \kappa_{(\beta,\gamma)}) \geq 0$ by definition and, hence, $\mu_{(\beta,\gamma)} > 0$ since $(\beta,\gamma) \notin D^*$, so that $\phi_{(\beta,\gamma)}$ is well-defined and belongs to Δ_{2I}. Moreover, if $(\beta,\gamma) \in \mathrm{cone}(\Phi) + D^*$, i.e. $(\beta,\gamma) = \mu(\beta',\gamma') - (\kappa,\kappa)$ for some $(\beta',\gamma') \in \Phi$, $\mu \in \mathbb{R}_+$, and $\kappa \in \mathbb{R}_+^I$, then $\kappa_{(\beta,\gamma)} \leq \kappa$ by definition and, hence, $\phi_{(\beta,\gamma)} \in \mathrm{cone}(\Phi) + D^*$. Now, let $(\beta,\gamma) \in K_\Phi^* \setminus D^*$. Since D^* is closed, there exists a sequence $(\beta^n,\gamma^n)_{n\in\mathbb{N}}$ such that $(\beta^n,\gamma^n) \in (\mathrm{cone}(\Phi) + D^*) \setminus D^*$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(\beta^n,\gamma^n) = (\beta,\gamma)$. Hence $\phi_{(\beta^n,\gamma^n)} \in (\mathrm{cone}(\Phi) + D^*) \cap \Delta_{2I}$ for all $n \in \mathbb{N}$ by definition and $\lim_{n\to\infty} \phi_{(\beta^n,\gamma^n)} = \phi_{(\beta,\gamma)}$ since $\phi_{(\cdot)}$ is continuous, so $\phi_{(\beta,\gamma)} \in \langle\Phi\rangle$. It follows that $(\beta,\gamma) = \mu_{(\beta,\gamma)}\phi_{(\beta,\gamma)} - (\kappa_{(\beta,\gamma)}, \kappa_{(\beta,\gamma)}) \in \mathrm{cone}(\langle\Phi\rangle) + D^*$.

To prove the second equality, first note that $\langle\Phi\rangle \subseteq \mathrm{cl}(\mathrm{cone}(\Phi) + D^*) \cap \Delta_{2I} = K_\Phi^* \cap \Delta_{2I}$ by definition. Conversely, let $(\beta,\gamma) \in K_\Phi^* \cap \Delta_{2I}$. Then as shown in the previous paragraph, we have $\phi_{(\beta,\gamma)} \in \langle\Phi\rangle$ since $(\beta,\gamma) \in K_\Phi^* \setminus D^*$. Moreover, $\phi_{(\beta,\gamma)} = (\beta,\gamma)$ since $(\beta,\gamma) \in \Delta_{2I}$, so $(\beta,\gamma) \in \langle\Phi\rangle$. □

By Lemma 10, for all $\Phi' \subseteq \Delta_{2I}$, we have $K_\Phi^* = K_{\Phi'}^*$ if and only if $\langle\Phi\rangle = \langle\Phi'\rangle$, which establishes the uniqueness claim.

A.3 Proof of Theorem 10

Clearly, if there exists a weakly constant linear functional $h : D \to \mathbb{R}$ representing F then F satisfies Pareto Indifference, IIA, Completeness, Mixture Continuity, Egalitarian Independence, and Egalitarian Non-Triviality. Conversely, assume F satisfies these axioms. First note that Lemma 1 holds since its proof does not rely on Independence. Next, we establish a weaker version of Lemma 2, reflecting the fact that F is no longer assumed to satisfy Independence but only Egalitarian Independence.

Lemma 11. For all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathcal{P}^I$, all $x, x' \in X$, all $y \in \hat{X}_{(U_i)_{i\in I}}$, and all $y' \in \hat{X}_{(U'_i)_{i\in I}}$ such that $U_i|_y^x = U'_i|_{y'}^{x'}$ for all $i \in I$, $x \succsim_{(U_i)_{i\in I}} y$ if and only if $x' \succsim_{(U'_i)_{i\in I}} y'$.

Proof. The proof is identical to that of Lemma 2, noting that $z \in \hat{X}_{(V_i)_{i\in I}} \cap \hat{X}_{(V'_i)_{i\in I}}$ in that proof since $y \in \hat{X}_{(U_i)_{i\in I}}$ and $y' \in \hat{X}_{(U'_i)_{i\in I}}$ and relying on Egalitarian Independence rather than Independence. □

Let

$$\hat{E} = \left\{ ((\min U_i|_y^x)_{i\in I}, (\max U_i|_y^x)_{i\in I}) \in \mathbb{R}^{2I} \,:\, (U_i)_{i\in I} \in \mathcal{P}^I, x \in X, y \in \hat{X}_{(U_i)_{i\in I}} \right\},$$
$$\hat{K} = \left\{ ((\min U_i|_y^x)_{i\in I}, (\max U_i|_y^x)_{i\in I}) \in \mathbb{R}^{2I} \,:\, (U_i)_{i\in I} \in \mathcal{P}^I, x \in X, y \in \hat{X}_{(U_i)_{i\in I}}, x \succsim_{(U_i)_{i\in I}} y \right\}.$$

$\hat{K}$ is a subset of the set K defined in the proof of Theorem 8, corresponding to the additional restriction that y must be egalitarian in $(U_i)_{i\in I}$. It is easy to see that $\hat{K} \subseteq \hat{E} = D$. We now establish analogues to Lemmas 5 and 6.

Lemma 12. For all $(U_i)_{i\in I} \in \mathcal{P}^I$ all $x \in X$, and all $y \in \hat{X}_{(U_i)_{i\in I}}$,

$$x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow ((\min U_i|_y^x)_{i\in I}, (\max U_i|_y^x)_{i\in I}) \in \hat{K}.$$

Proof. The "if" part holds by definition of $\hat{K}$. The "only if" part follows from Lemma 11. □

Lemma 13. $\hat{K}$ is a cone and $0 \in \hat{K}$.

Proof. First, by Pareto Indifference, $0 \in \hat{K}$. Second, the proof that $\hat{K}$ is a cone is identical to the proof that K is a cone in Lemma 6, noting that $y \in \hat{X}_{(U_i)_{i\in I}}$ by definition in that proof and relying on Egalitarian Independence and Lemma 12 rather than Independence and Lemma 5. □

Next, we establish further properties of $\hat{K}$, relying on Completeness and Egalitarian Non-Triviality.

Lemma 14. There exists a (unique) $\sigma \in \{-1, 1\}$ such that for all $c \in \mathbb{R}$, $c \in \hat{K}$ if and only if $\sigma c \geq 0$.

Proof. Uniqueness is obvious. Regarding existence, by Lemma 13, it suffices to show that either $1 \in \hat{K}$ or $-1 \in \hat{K}$ but not both. Suppose that both $1 \in \hat{K}$ and $-1 \in \hat{K}$. Then $(c, c) \in \hat{K}$ for all $c \in \mathbb{R}$ by Lemma 13. Hence we have $x \sim_{(U_i)_{i\in I}} y$ for all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in \hat{X}_{(U_i)_{i\in I}}$, contradicting Egalitarian Non-Triviality. Suppose that neither $1 \in \hat{K}$ nor $-1 \in \hat{K}$. Let Y be an affine basis of Y and let $x, y \in Y$. Define $u \in P$ by

$$u(x) = 1, \qquad\qquad u(z) = 0 \text{ for all } z \in Y \setminus \{x\}.$$

For all $i \in I$, let $U_i = \{u\}$. Then $U_i \in \mathcal{P}$, $U_i|_y^x = 1$, and $U_i|_x^y = -1$. Moreover, $x, y \in \hat{X}_{(U_i)_{i\in I}}$. Hence by Lemma 12 and since $\succsim_{(U_i)_{i\in I}}$ is complete, we have both $y \succ_{(U_i)_{i\in I}} x$ and $x \succ_{(U_i)_{i\in I}} y$, a contradiction. □

Lemma 15. For all $(s, t) \in \hat{K}$ and all $c \in \mathbb{R}_-$, $(s, t) - \sigma c \in \hat{K}$.

Proof. Let Y be an affine basis of X and let $x, y, z \in Y$. For all $i \in I$, define $u_i, v_i \in P$ by

$$u_i(x) = s_i, \qquad u_i(y) = \sigma c, \qquad u_i(w) = 0 \text{ for all } w \in Y \setminus \{x, y\},$$
$$v_i(x) = t_i, \qquad v_i(y) = \sigma c, \qquad v_i(w) = 0 \text{ for all } w \in Y \setminus \{x, y\}.$$

Let $U_i = \text{conv}(\{u_i, v_i\})$. Then $U_i \in \mathcal{P}$, $U_i|_z^x = [s_i, t_i]$, $U_i|_y^z = -\sigma c$, and $U_i|_y^x = [s_i - \sigma c, t_i - \sigma c]$. Moreover, $y, z \in \hat{X}_{(U_i)_{i\in I}}$. By Lemmas 12 and 14, it follows that $x \succsim_{(U_i)_{i\in I}} z \succsim_{(U_i)_{i\in I}} y$ and, hence, $x \succsim_{(U_i)_{i\in I}} y$ since $\succsim_{(U_i)_{i\in I}}$ is transitive, so that $(s,t) - \sigma c \in \hat{K}$. □

Lemma 16. For all $(s,t) \in D$, there exist $c, c' \in \mathbb{R}$ such that $(s,t) - c \in \hat{K}$ and $(s,t) - c' \notin \hat{K}$.

Proof. Let $\hat{c} \in \mathbb{R}$. Let Y be an affine basis of X and let $x, y, z \in Y$. For all $i \in I$, define $u_i, v_i \in P$ by

$$u_i(x) = s_i, \qquad u_i(y) = \hat{c}, \qquad u_i(w) = 0 \text{ for all } w \in Y \setminus \{x, y\},$$
$$v_i(x) = t_i, \qquad v_i(y) = \hat{c}, \qquad v_i(w) = 0 \text{ for all } w \in Y \setminus \{x, y\}.$$

Let $U_i = \text{conv}(\{u_i, v_i\})$. Then $U_i \in \mathcal{P}$ and $U_i|_z^{\lambda x + (1-\lambda)y} = [\lambda s_i + (1-\lambda)\hat{c}, \lambda t_i + (1-\lambda)\hat{c}]$ for all $\lambda \in [0,1]$. Moreover, $y, z \in \hat{X}_{(U_i)_{i\in I}}$. Hence for all $\lambda \in (0,1]$, we have

$$\lambda x + (1-\lambda)y \succsim_{(U_i)_{i\in I}} z \Leftrightarrow \lambda(s,t) + (1-\lambda)\hat{c} \in \hat{K} \Leftrightarrow (s,t) + \frac{(1-\lambda)}{\lambda}\hat{c} \in \hat{K},$$

where the first equivalence follows from Lemma 12 and the second one from Lemma 13.

Suppose that $(s,t) - c \notin \hat{K}$ for all $c \in \mathbb{R}$. Then $z \succ_{(U_i)_{i\in I}} \lambda x + (1-\lambda)y$ for all $\lambda \in (0,1]$ since $\succsim_{(U_i)_{i\in I}}$ is complete and, hence, $z \succsim_{(U_i)_{i\in I}} y$ since $\succsim_{(U_i)_{i\in I}}$ is mixture continuous. By Lemma 12, it follows that $-\hat{c} \in \hat{K}$ for all $\hat{c} \in \mathbb{R}$, contradicting Lemma 14. Hence $(s,t) - c \in \hat{K}$ for some $c \in \mathbb{R}$.

Suppose that $(s,t) - c' \in \hat{K}$ for all $c' \in \mathbb{R}$. Then $\lambda x + (1-\lambda)y \succsim_{(U_i)_{i\in I}} z$ for all $\lambda \in (0,1]$ and, hence, $y \succsim_{(U_i)_{i\in I}} z$ since $\succsim_{(U_i)_{i\in I}}$ is mixture continuous. By Lemma 12, it follows that $\hat{c} \in \hat{K}$ for all $\hat{c} \in \mathbb{R}$, again contradicting Lemma 14. Hence $(s,t) - c' \notin \hat{K}$ for some $c' \in \mathbb{R}$. □

Now, define the functional $h : D \to \mathbb{R}$ by for all $(s,t) \in D$,

$$h(s,t) = \sup\left\{c \in \mathbb{R} : (s,t) - \sigma c \in \hat{K}\right\}.$$

By Lemmas 15 and 16, h is well-defined. We now show that $\sigma h(s,t)$ is the "egalitarian equivalent" of (s,t) in the sense that an alternative with individual utility intervals $([s_i, t_i])_{i\in I}$ is indifferent to an egalitarian alternative with individual utility level c if and only if $c = \sigma h(s,t)$.

Lemma 17. For all $(U_i)_{i\in I} \in \mathcal{P}^I$, all $x \in X$, and all $y \in \hat{X}_{(U_i)_{i\in I}}$, $x \sim_{(U_i)_{i\in I}} y$ if and only if $u_i(y) = \sigma h((\min U_i|_{\{x\}})_{i\in I}, (\max U_i|_{\{x\}})_{i\in I})$ for all $i \in I$ and all $u_i \in U_i$.

Proof. Let $(s,t) = ((\min U_i|_{\{x\}})_{i\in I}, (\max U_i|_{\{x\}})_{i\in I})$. Since the affine dimension of X is at least 2, there exists $z \in X$ such that (x, y, z) are affinely independent. Let Y be an affine basis of X containing $\{x, y, z\}$. For all $i \in I$, define $v_i, v'_i \in P$ by

$$v_i(x) = s_i, \quad v_i(y) = \sigma(h(s,t)+1), \quad v_i(z) = \sigma(h(s,t)-1), \quad v_i(w) = 0 \text{ for all } w \in Y \setminus \{x,y,z\},$$
$$v'_i(x) = t_i, \quad v'_i(y) = \sigma(h(s,t)+1), \quad v'_i(z) = \sigma(h(s,t)-1), \quad v'_i(w) = 0 \text{ for all } w \in Y \setminus \{x,y,z\}.$$

Let $V_i = \text{conv}(\{v_i, v'_i\})$. Then $V_i \in \mathcal{P}$, $V_i|_{\{x\}} = [s_i, t_i]$, $V_i|_{\{0.5y+0.5z\}} = \{\sigma h(s,t)\}$, and $V_i|_{\lambda y+(1-\lambda)z}^x = [s_i - \sigma(h(s,t) + 2\lambda - 1), t_i - \sigma(h(s,t) + 2\lambda - 1)]$ for all $\lambda \in [0,1]$. By Lemma 15, it follows that $V_i|_{\lambda y+(1-\lambda)z}^x \in \hat{K}$ for all $\lambda \in [0, 0.5)$ whereas $V_i|_{\lambda y+(1-\lambda)z}^x \notin \hat{K}$ for all $\lambda \in (0.5, 1]$. Hence $x \succsim_{(V_i)_{i\in I}} \lambda y + (1-\lambda)z$ for all $\lambda \in [0, 0.5)$ whereas $\lambda y + (1-\lambda)z \succ_{(V_i)_{i\in I}} x$ for all $\lambda \in (0.5, 1]$ since $\succsim_{(V_i)_{i\in I}}$ is complete. Hence $x \sim_{(V_i)_{i\in I}} 0.5y + 0.5z$ since $\succsim_{(V_i)_{i\in I}}$ is mixture continuous and, hence, $x \sim_{(V_i)_{i\in I}} y$ by Lemma 1, establishing the "if" part.

For the "only if" part, suppose $x' \sim_{(U_i')_{i\in I}} y'$ for some $(U_i')_{i\in I} \in \mathcal{P}^I$ and some $x', y' \in X$ such that $U_i'|_{\{x'\}} = [s_i, t_i]$ and $U_i'|_{\{y'\}} = \{c\} \neq \{\sigma h(s,t)\}$ for all $i \in I$. Since the affine dimension of X is at least 2, there exists $z' \in X$ such that (x', y', z') are affinely independent. Let Y' be an affine basis of X containing $\{x', y', z'\}$ For all $i \in I$, define $v_i, v_i' \in P$ by

$$
\begin{aligned}
&v_i(x') = s_i, \qquad v_i(y') = \sigma h(s,t), \qquad v_i(z') = c, \qquad v_i(w) = 0 \text{ for all } w \in Y' \setminus \{x', y', z'\},\\
&v_i'(x') = t_i, \qquad v_i'(y') = \sigma h(s,t), \qquad v_i'(z') = c, \qquad v_i'(w) = 0 \text{ for all } w \in Y' \setminus \{x', y', z'\}.
\end{aligned}
$$

Let $V_i = \operatorname{conv}(\{v_i, v_i'\})$. Then $V_i \in \mathcal{P}$, $V_i|_{\{x'\}} = [s_i, t_i]$, $V_i|_{\{y'\}} = \{\sigma h(s,t)\}$, and $V_i|_{\{z'\}} = \{c\}$. Moreover, $y, z \in \hat{X}_{(V_i)_{i\in I}}$. It follows that $y' \sim_{(V_i)_{i\in I}} x' \sim_{(V_i)_{i\in I}} z'$ by Lemma 1 and, hence, $y' \sim_{(V_i)_{i\in I}} z'$ since $\succsim_{(V_i)_{i\in I}}$ is transitive. Hence both $\sigma h(s,t) - c \in \hat{K}$ and $c - \sigma h(s,t) \in \hat{K}$, contradicting Lemma 14. $\square$

The next two lemmas show that h represents F and is weakly constant linear, establishing the main result.

Lemma 18. For all $(U_i)_{i\in I} \in \mathcal{P}^I$ and all $x, y \in X$,

$$
x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow h((\min U_i|_{\{x\}})_{i\in I}, (\max U_i|_{\{x\}})) \geq h((\min U_i|_{\{y\}})_{i\in I}, (\max U_i|_{\{y\}})).
$$

Proof. Let $c = \sigma h((\min U_i|_{\{x\}})_{i\in I}, (\max U_i|_{\{x\}}))$ and $c' = \sigma h((\min U_i|_{\{y\}})_{i\in I}, (\max U_i|_{\{y\}}))$, so we need to show that $x \succsim_{(U_i)_{i\in I}} y$ if and only if $\sigma c \geq \sigma c'$. Since the affine dimension of X is at least 2, there exists $z \in X$ such that (x, y, z) are affinely independent. For all $u \in P$, define $v_u, v_u' \in P$ by

$$
\begin{aligned}
&v_u(x) = u(x), \qquad v_u(y) = u(y), \qquad v_u(z) = c, \qquad v_u(w) = 0 \text{ for all } w \in Y \setminus \{x, y, z\},\\
&v_u'(x) = c', \qquad v_u'(y) = u(y), \qquad v_u'(z) = c, \qquad v_u'(w) = 0 \text{ for all } w \in Y \setminus \{x, y, z\}.
\end{aligned}
$$

For all $i \in I$, let $V_i = \{v_u : u \in U_i\}$ and $V_i' = \{v_u' : u \in U_i\}$. Then $V_i, V_i' \in \mathcal{P}$, $V_i|_{\{x,y\}} = U_i|_{\{x,y\}}$, and $V_i'|_{\{y,z\}} = V_i|_{\{y,z\}}$. Moreover, we have $z \in \hat{X}_{(V_i)_{i\in I}}$ and $x \sim z_{(V_i)_{i\in I}}$ as well as $x \in \hat{X}_{(V_i')_{i\in I}}$ and $y \sim_{(V_i')_{i\in I}} x$ by Lemma 17. Hence

$$
x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow x \succsim_{(V_i)_{i\in I}} y \Leftrightarrow z \succsim_{(V_i)_{i\in I}} y \Leftrightarrow z \succsim_{(V_i')_{i\in I}} y \Leftrightarrow z \succsim_{(V_i')_{i\in I}} z' \Leftrightarrow \sigma c \geq \sigma c',
$$

where the first and third equivalences follow from Lemma 1, the second and fourth ones from transitivity of $\succsim_{(V_i)_{i\in I}}$ and $\succsim_{(V_i)_{i\in I}}$, respectively, and the fifth one from Lemmas 12 and 14. $\square$

Lemma 19. *h is weakly constant linear.*

Proof. First, by Lemma 17, we have $h(c) = \sigma c$ for all $c \in \mathbb{R}$, so h is weakly normalized. Moreover, noting that $(s,t) - \sigma c = (s,t) + c' - \sigma(c + \sigma c')$ for all $(s,t) \in D$ and all $c, c' \in \mathbb{R}$, we have $h((s,t) + c') = h(s,t) + \sigma c' = h(s,t) + h(c')$ by definition of h, so h is weakly constant additive. Finally, by Lemma 13, we have $(s,t) - \sigma c \in \hat{K}$ if and only if $\mu(s,t) - \sigma \mu c \in \hat{K}$ for all $(s,t) \in D$, all $c \in \mathbb{R}$, and all $\mu \in \mathbb{R}_{++}$. It follows that $h(\mu(s,t)) = \mu h(s,t)$ by definition of h, so h is positively homogeneous and, hence, weakly constant linear. $\square$

Finally, the following lemma establishes the uniqueness claim.

Lemma 20. If a weakly normalized and positively homogeneous functional $h' : D \to \mathbb{R}$ represents F then $h' = h$.

Proof. Since h' represents F, we have $h'(s,t) = h'(\sigma h(s,t))$ for all $(s,t) \in D$ by Lemma 17. Moreover, since h' is weakly normalized and positively homogeneous, we have $h'(c) = \sigma h(c)$ for all $c \in \mathbb{R}$ by Lemma 14. Hence $h'(s,t) = h(s,t)$ for all $(s,t) \in D$. $\square$

A.4 Proof of Theorem 9

Clearly, if there exists a non-empty, compact, and convex set $\Phi \subseteq \hat{\Delta}_{2I}$ representing F then F satisfies Pareto Indifference, IIA, Completeness, Egalitarian Independence, Inequality Aversion, Mixture Continuity, and Egalitarian Non-Triviality. Conversely, assume F satisfies these axioms. We first establish an analogue to Lemma 3, using Inequality Aversion.

Lemma 21. For all $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathcal{P}^I$ all $x \in X$, and all $y \in \hat{X}_{(U_i)_{i\in I}} \cap \hat{X}_{(U'_i)_{i\in I}}$ such that, for all $i \in I$, $\lambda_i \min U_i|^x_y + (1-\lambda_i)\max U_i|^x_y \in U'_i|^x_y \subseteq U_i|^x_y$ for some $\lambda_i \in (0,1)$, if $x \succsim_{(U_i)_{i\in I}} y$ then $x \succsim_{(U'_i)_{i\in I}} y$.

Proof. The proof is identical to that of Lemma 3, noting that $z \in \hat{X}_{(V_i)_{i\in I}} \cap \hat{X}_{(V'_i)_{i\in I}}$ (since $y \in \hat{X}_{(U_i)_{i\in I}} \cap \hat{X}_{(U'_i)_{i\in I}}$) and $x \sim_{(V_i)_{i\in I}} y$ (by Lemma 1) in that proof and relying on Inequality Aversion and Lemma 11 rather than Independence and Lemma 2. $\square$

Define the set $\hat{K}$ as in the proof of Theorem 10.

Lemma 22. $\hat{K}$ is a non-empty, closed, and convex cone.

Proof. By Lemma 13, $\hat{K}$ is a non-empty cone. To prove that $\hat{K}$ is convex, let $(s,t), (s',t') \in \hat{K}$ and suppose $\lambda(s,t) + (1-\lambda)(s',t') \notin \hat{K}$ for some $\lambda \in (0,1)$. Let $Y, x, y, z, (U_i)_{i\in I}$ be as in the proof that K is convex in Lemma 6 and note that $z \in \hat{X}_{(U_i)_{i\in I}}$ by definition. Then $\lambda x + (1-\lambda)y \not\succsim_{(U_i)_{i\in I}} z$ by Lemma 12. Since $\succsim_{(U_i)_{i\in I}}$ is complete and mixture continuous, it follows that there exist $\underline{\lambda}, \overline{\lambda} \in [0,1]$, with $\underline{\lambda} < \lambda < \overline{\lambda}$, such that $z \succ_{(U_i)_{i\in I}} \lambda' x + (1-\lambda')y$ for all $\lambda' \in (\underline{\lambda}, \overline{\lambda})$ and $\underline{\lambda} x + (1-\underline{\lambda})y \sim_{(U_i)_{i\in I}} z \sim_{(U_i)_{i\in I}} \overline{\lambda} x + (1-\overline{\lambda})y$. Hence $0.5(\underline{\lambda} + \overline{\lambda})x + (1 - 0.5(\underline{\lambda} + \overline{\lambda}))y \succsim_{(U_i)_{i\in I}} z$ by Inequality Aversion and since $\succsim_{(U_i)_{i\in I}}$ is transitive, a contradiction. The proof that $\hat{K}$ is a closed is identical to the proof that K is closed in Lemma 6, noting that $z \in \hat{X}_{(U_i)_{i\in I}}$ by definition in that proof and relying on Lemma 12 rather than Lemma 5. $\square$

Lemma 23. For all $(s,t), (s',t') \in D$ such that $[s'_i, t'_i] \subseteq [s_i, t_i]$ for all $i \in I$, if $(s,t) \in \hat{K}$ then $(s',t') \in \hat{K}$.

Proof. The proof is identical to that of Lemma 7, relying on Lemmas 21 and 22 rather than Lemmas 3 and 6. $\square$

Define the polar C^* of a cone C in $\mathbb{R}^{2I}$ as in the proof of Theorem 8.

Lemma 24. $D^* = \{-(\kappa,\kappa) : \kappa \in \mathbb{R}^I_+\}$ and $\hat{K}^* = \text{cone}(\hat{K}^* \cap \Delta_{2I}) + D^*$.

Proof. The former equality is proved in Lemma 8. The proof of the latter equality is identical to that of the analogue equality in Lemma 8, relying on Lemma 23 rather than Lemma 7. $\square$

Lemma 25. $(\sum_{i\in I} \beta_i - \gamma_i)(\sum_{i\in I} \beta'_i - \gamma'_i) > 0$ for all $(\beta,\gamma), (\beta',\gamma') \in \hat{K}^* \cap \Delta_{2I}$.

Proof. Since $\hat{K}^*$ and Δ_{2I} are convex, it suffices to show that $\sum_{i\in I}\beta_i - \gamma_i \neq 0$ for all $(\beta,\gamma)\in\hat{K}^*\cap\Delta_{2I}$. So suppose that $\sum_{i\in I}\beta_i-\gamma_i = 0$ for some $(\beta,\gamma)\in\hat{K}^*\cap\Delta_{2I}$. Then by definition of $\hat{K}^*$, for all $c\in\mathbb{R}$, $(0,1)-c\in\hat{K}^*$ implies $0\le\sum_{i\in I}-c\beta_i-(1-c)\gamma_i = -\sum_{i\in I}\gamma_i = -0.5c$, which is impossible. Hence $(0,1)-c\notin\hat{K}$ for all $c\in\mathbb{R}$, contradicting Lemma 16. □

Given a subset Φ of $\hat{\Delta}_{2I}$, define the functional $h_\Phi : D\to\mathbb{R}$ by for all $(s,t)\in D$,

$$h_\Phi(s,t) = \min_{(\beta,\gamma)\in\Phi}\frac{\sum_{i\in I}\beta_i s_i-\gamma_i t_i}{\left|\sum_{i\in I}\beta_i-\gamma_i\right|}.$$

Clearly, h_Φ is weakly normalized and positively homogeneous.

Lemma 26. For all $\Phi\subseteq\hat{\Delta}_{2I}$, h_Φ is weakly constant additive if and only if $(\sum_{i\in I}\beta_i-\gamma_i)(\sum_{i\in I}\beta_i'-\gamma_i')>0$ for all $(\beta,\gamma),(\beta',\gamma')\in\Phi$.

Proof. For all $(s,t)\in D$ and all $c\in\mathbb{R}$, we have

$$h_\Phi((s,t)+c) = \min_{(\beta,\gamma)\in\Phi}\left(\frac{\sum_{i\in I}\beta_i s_i-\gamma_i t_i}{\left|\sum_{i\in I}\beta_i-\gamma_i\right|}+\frac{\sum_{i\in I}\beta_i-\gamma_i}{\left|\sum_{i\in I}\beta_i-\gamma_i\right|}c\right).$$

If $(\sum_{i\in I}\beta_i-\gamma_i)(\sum_{i\in I}\beta_i'-\gamma_i')>0$ for all $(\beta,\gamma),(\beta',\gamma')\in\Phi$ then there exists a $\tau\in\{-1,1\}$ such that

$$\frac{\sum_{i\in I}\beta_i-\gamma_i}{\left|\sum_{i\in I}\beta_i-\gamma_i\right|}=\tau$$

for all $(\beta,\gamma)\in\Phi$ and, hence,

$$h_\Phi((s,t)+c) = \min_{(\beta,\gamma)\in\Phi}\left(\frac{\sum_{i\in I}\beta_i s_i-\gamma_i t_i}{\left|\sum_{i\in I}\beta_i-\gamma_i\right|}\right)+\tau c = h_\Phi(s,t)+h_\Phi(c),$$

so that h_Φ is weakly constant additive. Conversely, if $\sum_{i\in I}\beta_i-\gamma_i>0$ and $\sum_{i\in I}\beta_i'-\gamma_i'<0$ for some $(\beta,\gamma),(\beta',\gamma')\in\Phi$ then $h_\Phi(1)=h_\Phi(-1)=-1$ and, hence, $h_\Phi(1)+h_\Phi(-1)\neq 0 = h_\Phi(0)$, so that h_Φ is not weakly constant additive. □

Lemma 27. A set $\Phi\subseteq\hat{\Delta}_{2I}$ represents F if and only if $\mathrm{cl}(\mathrm{cone}(\Phi)+D^*)=\hat{K}^*$.

Proof. Let h be the unique weakly constant linear functional representing F as per Theorem 10. Then since h_Φ is weakly normalized and positively homogeneous, Φ represents F if and only if $h_\Phi = h$ by Lemma 20. Moreover, we have $\mathrm{cl}(\mathrm{cone}(\Phi)+D^*) = K_\Phi^*$ (Rockafellar, 1970, Corollary 16.4.2), so that $\mathrm{cl}(\mathrm{cone}(\Phi)+D^*)=\hat{K}^*$ if and only if $K_\Phi=\hat{K}$. Hence it suffices to show that $h_\Phi = h$ if and only if $K_\Phi=\hat{K}$. To this end, by Lemmas 25 and 26, we can assume that $(\sum_{i\in I}\beta_i-\gamma_i)(\sum_{i\in I}\beta_i'-\gamma_i')>0$ for all $(\beta,\gamma),(\beta',\gamma')\in\Phi$. We can further assume that

$$\frac{\sum_{i\in I}\beta_i-\gamma_i}{\left|\sum_{i\in I}\beta_i-\gamma_i\right|}=\sigma$$

for all $(\beta,\gamma)\in\Phi$, where σ is defined in Lemma 14, for otherwise we can have neither $h_\Phi = h$ nor

$K_\Phi = \hat{K}$. Hence for all $(s,t) \in D$ and all $c \in \mathbb{R}$, we have

$$
\begin{aligned}
(s,t) - \sigma c \in K_\Phi &\Leftrightarrow \left[\forall(\beta,\gamma) \in \Phi, \sum_{i\in I} \beta_i(s_i - \sigma c) - \gamma_i(t_i - \sigma c) \geq 0\right] \\
&\Leftrightarrow \left[\forall(\beta,\gamma) \in \Phi, \frac{\sum_{i\in I} \beta_i s_i - \gamma_i t_i}{\left|\sum_{i\in I} \beta_i - \gamma_i\right|} \geq c\right] \\
&\Leftrightarrow \min_{(\beta,\gamma)\in\Phi} \frac{\sum_{i\in I} \beta_i s_i - \gamma_i t_i}{\left|\sum_{i\in I} \beta_i - \gamma_i\right|} \geq c \\
&\Leftrightarrow h_\Phi(s,t) \geq c.
\end{aligned}
$$

Hence if $h_\Phi = h$ then for all $(s,t) \in D$, we have

$$
(s,t) \in \hat{K} \Leftrightarrow h(s,t) \geq 0 \Leftrightarrow h_\Phi(s,t) \geq 0 \Leftrightarrow (s,t) \in K_\Phi,
$$

so that $\hat{K} = K_\Phi$. Conversely, if $\hat{K} = K_\Phi$ then for all $(s,t) \in D$, we have

$$
\begin{aligned}
h(s,t) &= \sup\left\{c \in \mathbb{R} : (s,t) - \sigma c \in \hat{K}\right\} \\
&= \sup\left\{c \in \mathbb{R} : (s,t) - \sigma c \in \hat{K}_\Phi\right\} \\
&= \sup\left\{c \in \mathbb{R} : h_\Phi(s,t) \geq c\right\} \\
&= h_\Phi(s,t),
\end{aligned}
$$

so that $h_\Phi = h$. □

Let $\Phi = \hat{K}^* \cap \Delta_{2I}$. Then Φ is ccompact and convex since $\hat{K}^*$ is closed and convex and Δ_{2I} is compact and convex. Moreover, $K \subset \hat{\Delta}_{2I}$ by Lemma 25 and $K^* = \text{cone}(\Phi) + D^* = \text{cl}(\text{cone}(\Phi) + D^*)$ by Lemma 8 and since K^* is closed. Since K^* is non-empty, it follows that Φ is non-empty as well. This establishes the main result by Lemma 27. The uniqueness claim then follows from Lemma 10 as in the proof of Theorem 8.

A.5 Proofs of Theorems 1–6

Proof of Theorem 1. Follows immediately from Theorem 7. □

Proof of Theorem 2. It suffices to show that in Theorem 8, F satisfies Pareto Preference if and only if $\gamma = 0$ for all $(\beta,\gamma) \in \Phi$ (the uniqueness claim then follows straightforwardly from the definition of $\langle\Phi\rangle$). Clearly, if $\gamma = 0$ for all $(\beta,\gamma) \in \Phi$ then F satisfies Pareto Preference. Conversely, assume F satisfies Pareto Preference. Let Y be an affine basis of X and $x, y \in Y$. For all $i \in I$, define $u_i, v_i \in P$ by

$$
\begin{aligned}
u_i(x) &= 0, \qquad & u_i(z) &= 0 \text{ for all } z \in Y \setminus \{x\}, \\
v_i(x) &= 1, \qquad & v_i(z) &= 0 \text{ for all } z \in Y \setminus \{x\}.
\end{aligned}
$$

Let $U_i = \text{conv}(\{u_i, v_i\})$. Then $U_i \in \mathcal{P}$. Moreover, $x \succsim_{(U_i)_{i\in I}} y$ by Pareto Preference and, hence, $-\sum_{i\in I} \gamma_i \geq 0$ for all $(\beta,\gamma) \in \Phi$. Since $\Phi \geq 0$, it follows that $\gamma = 0$ for all $(\beta,\gamma) \in \Phi$. □

Proof of Theorem 5. It suffices to show that in Theorem 10, F satisfies Pareto Preference if and only if h is monotonic. Clearly, if h is monotonic then F satisfies Pareto Preference. Conversely, assume F satisfies Pareto Preference and let $(s,t),(s',t') \in D$ such that $s \geq s'$ and $t \geq t'$. Llet Y be an affine basis of X, and let $x, y \in Y$. For all $i \in I$, define $u_i, v_i \in P$ by

$$\begin{aligned} &u_i(x) = s_i, && u_i(y) = s_i', && u_i(z) = 0 \text{ for all } z \in Y \setminus \{x,y\}, \\ &v_i(x) = t_i, && v_i(y) = t_i' && v_i(z) = 0 \text{ for all } z \in Y \setminus \{x,y\}. \end{aligned}$$

Let $U_i = \operatorname{conv}(\{u_i, v_i\})$. Then $U_i \in \mathcal{P}$. Moreover, $x \succsim_{(U_i)_{i\in I}} y$ by Pareto Preference and, hence, $h(s,t) \geq h(s',t')$, so that h is monotonic. □

Proof of Theorem 3. It suffices to show that in Theorem 2, F satisfies Pareto Preference if and only if $\gamma = 0$ for all $(\beta,\gamma) \in \Phi$ (the uniqueness claim then follows straightforwardly from the definition of $\langle\Phi\rangle$). Clearly, if $\gamma = 0$ for all $(\beta,\gamma) \in \Phi$ then F satisfies Pareto Preference. Conversely, assume F satisfies Pareto Preference. Let Y be an affine basis of X and $x, y \in Y$. For all $i \in I$, define $u_i, v_i \in P$ by

$$\begin{aligned} &u_i(x) = 0, && u_i(z) = 0 \text{ for all } z \in Y \setminus \{x\}, \\ &v_i(x) = 1, && v_i(z) = 0 \text{ for all } z \in Y \setminus \{x\}. \end{aligned}$$

Let $U_i = \operatorname{conv}(\{u_i, v_i\})$. Then $U_i \in \mathcal{P}$. Moreover, $x \succsim_{(U_i)_{i\in I}} y$ by Pareto Preference and, hence,

$$\min_{(\beta,\gamma)\in\Phi} \frac{-\sum_{i\in I}\gamma_i}{\left|\sum_{i\in I}\beta_i - \gamma_i\right|} \geq 0.$$

Since $\Phi \geq 0$, it follows that $\gamma = 0$ for all $(\beta,\gamma) \in \Phi$. □

Proof of Theorem 4. It is obvious that (ii) implies (i). Conversely, assume that (i) holds. Then there exists a unique non-empty, compact, and convex set $\Theta \subseteq \Delta_I$ representing F^* as per Theorem 2. Moreover, since F^* satisfies Pareto Preference, so does $F^\wedge$ by Consistency and, hence, there exists a unique constant linear and monotonic functional h representing $F^\wedge$ as per Theorem 5. Let Y be an affine basis of X, let $x, y, z \in Y$, and let $(U_i)_{i\in I} \in \mathcal{P}^I$. For all $u \in P$, define $v_u \in P$ by

$$v_u(x) = u(x), \quad v_u(y) = u_{\Theta,(U_i)_{i\in I}}(x) + 1, \quad v_u(z) = u_{\Theta,(U_i)_{i\in I}}(x) - 1, \quad v_u(w) = 0 \text{ for all } w \in Y \setminus \{x,y,z\}.$$

Let $V_i = \{v_u : u \in U_i\}$. Then $V_i \in \mathcal{P}$, $V_i|_{\{x\}} = U_i|_{\{x\}}$ and $V_i|_{\{\lambda y+(1-\lambda)z\}} = \{u_{\Theta,(U_i)_{i\in I}}(x)+2\lambda-1\}$ for all $\lambda \in [0,1]$. It follows that $x \succsim^*_{(V_i)_{i\in I}} \lambda y+(1-\lambda)z$ for all $\lambda \in [0,0.5)$ whereas $x \not\succsim^*_{(V_i)_{i\in I}} \lambda y+(1-\lambda)z$ for all $\lambda \in (0.5,1]$. Hence $x \succsim^\wedge_{(V_i)_{i\in I}} \lambda y+(1-\lambda)z$ for all $\lambda \in [0,0.5)$ by Consistency whereas $\lambda y+(1-\lambda)z \succsim^\wedge_{(V_i)_{i\in I}} x$ for all $\lambda \in (0.5,1]$ by Egalitarian Default and, hence, $x \sim^\wedge_{(V_i)_{i\in I}} 0.5y + 0.5z$ since $\succsim^\wedge_{(V_i)_{i\in I}}$ is mixture continuous. We therefore have $h((\min U_i|_{\{x\}})_{i\in I}, (\max U_i|_{\{x\}})) = u_{\Theta,(U_i)_{i\in I}}(x)$ by Lemma 17, so that Θ represents $F^\wedge$ as per Theorem 3, establishing (ii). □

Proof of Theorem 6. It is obvious that (ii) implies (i). Conversely, assume that (i) holds. Then there exists a unique non-empty, compact, and convex set $\Theta \subseteq \Delta_I$ representing F^* as per Theorem 2. Moreover, since F^* satisfies Pareto Preference, so does $F^\wedge$ by Consistency and, hence, there exists a unique constant linear and monotonic functional h representing $F^\wedge$ as per Theorem 5. We then proceed as in the proof of Ghirardato et al. (2004)'s Lemma B.5. Define the functionals $\underline{h}, \overline{h} : D \to \mathbb{R}$ by for all $(s,t) \in \mathbb{R}$,

$$\underline{h}(s,t) = \min_{\theta \in \Theta} \sum_{i \in I} \theta_i s_i, \qquad \overline{h}(s,t) = \max_{\theta \in \Theta} \sum_{i \in I} \theta_i t_i.$$

By Egalitarian Dominance and Lemma 17, there exists a functional $g : \mathbb{R}^2 \to \mathbb{R}$ such that $h(s,t) = g(\underline{h}(s,t), \overline{h}(s,t))$ for all $(s,t) \in D$. Moreover, by Consistency and Lemma 17, we have $\underline{h}(s,t) \leq h(s,t) \leq \overline{h}(s,t)$ for all $(s,t) \in D$. Hence for all $(s,t) \in D$ such that $\underline{h}(s,t) < \overline{h}(s,t)$, there exists a unique $\alpha(s,t) \in [0,1]$ such that $h(s,t) = \alpha(s,t)\underline{h}(s,t) + (1-\alpha(s,t))\overline{h}(s,t)$, i.e.

$$\begin{aligned}\alpha(s,t) = \frac{h(s,t) - \overline{h}(s,t)}{\underline{h}(s,t) - \overline{h}(s,t)} &= -h\left(\frac{(s,t) - \overline{h}(s,t)}{\overline{h}(s,t) - \underline{h}(s,t)}\right) \\ &= -g\left(\underline{h}\left(\frac{(s,t) - \overline{h}(s,t)}{\overline{h}(s,t) - \underline{h}(s,t)}\right), \overline{h}\left(\frac{(s,t) - \overline{h}(s,t)}{\overline{h}(s,t) - \underline{h}(s,t)}\right)\right) = -g(-1,0),\end{aligned}$$

so that $\alpha(s,t)$ is independent of (s,t). Finally, for all $(s,t) \in D$ such that $\underline{h}(s,t) = \overline{h}(s,t)$, we trivially have $h(s,t) = \alpha\underline{h}(s,t) + (1-\alpha)\overline{h}(s,t)$ for all $\alpha \in [0,1]$. Setting $\alpha = -g(-1,0) \in [0,1]$ thus ensures that $u_{\Theta,\alpha,(U_i)_{i\in I}}$ represents $\succsim^{\wedge}_{(U_i)_{i\in I}}$ for all $(U_i)_{i\in I} \in \mathcal{P}^I$, establishing (ii). $\square$

A.6 Proofs of Propositions 1–5

Proof of Proposition 1. We only prove the first claim, the second one then follows trivially. We consider Theorems 8 and 9 simultaneously. Clearly, if $(\pi(\beta), \pi(\gamma)) \in \langle\Phi\rangle$ for all $(\beta,\gamma) \in \langle\Phi\rangle$ then F satisfies Anonymity. Convesely assume that F satisfies Anonymity. Then $(\pi(\beta), \pi(\gamma)) \in K$ (resp. $\hat{K}$) for all $(\beta,\gamma) \in K$ (resp. $\hat{K}$) by definition. Hence $(\pi(\beta), \pi(\gamma)) \in K^*_\Phi$ for all $(\beta,\gamma) \in K^*_\Phi$ by Lemma 9 (resp. Lemma 27). It follows that $(\pi(\beta), \pi(\gamma)) \in \langle\Phi\rangle$ for all $(\beta,\gamma) \in \langle\Phi\rangle$ by Lemma 10. $\square$

Proof of Proposition 2. We only prove the first claim, the second one then follows trivially. We consider Theorems 8 and 9 simultaneously. First note that since Φ is a convex subset of $\mathbb{R}^{2I}_+$, there exists $(\beta,\gamma) \in \Phi$ such that $\beta+\gamma \gg 0$ if and only if for all $i \in I$, there exists $(\beta^i,\gamma^i) \in \Phi$ such that $\beta^i_i + \gamma^i_i > 0$, i.e. $\beta^i_i \neq 0$ or $\gamma^i_i \neq 0$. So assume this holds. Let $i \in I$, let Y be an affine basis of X, and let $x, y \in Y$. Define $v_i, v'_i \in P$ by

$$v_i(x) = -1, \qquad v_i(z) = 0 \text{ for all } z \in Y \setminus \{x\},$$
$$v'_i(x) = 1, \qquad v'_i(z) = 0 \text{ for all } z \in Y \setminus \{x\}.$$

Let $U_i = \mathrm{conv}(\{v_i, v'_i\}) \in \mathcal{P}$ and for all $j \in I \setminus \{i\}$, let $U_j = \{0\} \in \mathcal{P}$. Then $\beta^i_i v_i(x) - \gamma^i_i v'_i(x) = -\beta^i_i - \gamma^i_i < 0 = v_i(y) = v'_i(y)$ whereas $u_j(x) = u_j(y) = 0$ for all $j \in I \setminus \{i\}$ and, hence, $x \nsim_{(U_j)_{j\in I}} y$ since $U_{\Phi,(U_j)_{j\in I}}$ (resp. $u_{\Phi,(U_j)_{j\in I}}$) represents $\succsim_{(U_j)_{j\in I}}$, so F satisfies Full Support. Conversely, assume that for some $i \in I$, we have $\beta_i = \gamma_i = 0$ for all $(\beta,\gamma) \in \Phi$. Let $(U_j)_{j\in I} \in \mathcal{P}^I$ and $x, y \in X$ be such that $u_j(x) = u_j(y)$ for all $j \in I \setminus \{i\}$ and all $u_j \in U_j$. Then $x \sim_{(U_j)_{j\in I}} y$ since $U_{\Phi,(U_j)_{j\in I}}$ (resp. $u_{\Phi,(U_j)_{j\in I}}$) represents $\succsim_{(U_j)_{j\in I}}$, so F violates Full Support. $\square$

Proof of Proposition 3. The "if" part of both statements is obvious. For the "only if" part, assume F satisfies Singleton Pareto Strict Preference. For the former statement, since Θ is convex, it suffices to show that for all $i \in I$, there exists $\theta \in \Theta$ such that $\theta_i > 0$. For the latter statement, we need to show that $\theta_i > 0$ for all $i \in I$ and all $\theta \in \Theta$. Let $i \in I$, let Y be an affine basis of X, and let $x, y \in Y$. Define $u_i \in P$ by

$$u_i(x) = 1, \qquad\qquad u_i(z) = 0 \text{ for all } z \in Y \setminus \{x\}.$$

Let $U_i = \{u_i\} \in \mathcal{P}$ and for all $j \in I \setminus \{i\}$, let $U_j = \{0\} \in \mathcal{P}$. Then $x \succ_{(U_j)_{j\in I}} y$ by Singleton Pareto Strict Preference. Hence since $U_{\Theta,(U_j)_{j\in I}}$ (resp. $u_{\Theta,(U_j)_{j\in I}}$) represents $\succsim_{(U_j)_{j\in I}}$, we must have $\theta_i > 0$ for some (resp. all) $\theta \in \Theta$. □

Proof of Proposition 4. For the first claim, clearly, if Θ is a singleton then F satisfies Singleton Completeness. Conversely, assume there exists $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$. Then there exists $s \in \mathbb{R}^I$ such that $\sum_{i\in I} \theta_i s_i < 0 < \sum_{i\in I} \theta'_i s_i$. Let Y be an affine basis of X and $x, y \in Y$. For all $i \in I$, define $u_i \in P$ by

$$u_i(x) = s_i, \qquad\qquad u_i(z) = 0 \text{ for all } z \in Y \setminus \{x\}.$$

Then $\sum_{i\in I} \theta_i u_i(x) < \sum_{i\in I} \theta_i u_i(y)$ whereas $\sum_{i\in I} \theta'_i u_i(x) > \sum_{i\in I} \theta'_i u_i(y)$, so that neither $x \succsim_{(\{u_i\})_{i\in I}} y$ nor $y \succsim_{(\{u_i\})_{i\in I}} x$. Hence F violates Singleton Completeness.

For the second claim, clearly, if Θ is a singleton then F satisfies Singleton Independence. Conversely, assume there exist $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$. Then there exists $s \in \mathbb{R}^I$ such that $\sum_{i\in I} \theta_i s_i < 0 < \sum_{i\in I} \theta'_i s_i$. Let Y be an affine basis of X and $x, y \in Y$. For all $i \in I$, define $u_i \in P$ by

$$u_i(x) = s_i, \qquad u_i(y) = -s_i, \qquad u_i(z) = 0 \text{ for all } z \in Y \setminus \{x, y\}.$$

Then $\sum_{i\in I} \theta_i u_i(x) < 0$ and $\sum_{i\in I} \theta'_i u_i(y) < 0$ whereas $\sum_{i\in I} \theta''_i (0.5u_i(x) + 0.5u_i(y)) = 0$ for all $\theta'' \in \Delta_I$, so that both $0.5x + 0.5y \succ_{(\{u_i\})_{i\in I}} x$ and $0.5x + 0.5y \succ_{(\{u_i\})_{i\in I}} y$. Hence F violates Singleton Independence. □

Proof of Proposition 5. Assume that F satisfies CU. Let $(U_i)_{i\in I} \in \mathcal{P}^I$, $x, y, z \in X$, and $\lambda \in (0, 1)$. If $x \succsim_{(U_i)_{i\in I}} y$ then, letting $U'_i = \{\lambda u_i + (1-\lambda)u_i(z) : u_i \in U_i\} \in \mathcal{P}$ for all $i \in I$, we have $x \succsim_{(U'_i)_{i\in I}} y$ by CU and, hence, $\lambda x + (1-\lambda)z \succsim_{(U_i)_{i\in I}} \lambda y + (1-\lambda)z$ by Lemma 1. Conversely, if $\lambda x + (1-\lambda)z \succsim_{(U_i)_{i\in I}} \lambda y + (1-\lambda)z$ then, letting $U'_i = \{\frac{1}{\lambda}u_i - \frac{1-\lambda}{\lambda}u_i(z) : u_i \in U_i\} \in \mathcal{P}$ for all $i \in I$, we have $\lambda x + (1-\lambda)z \succsim_{(U'_i)_{i\in I}} \lambda y + (1-\lambda)z$ by CU and, hence, $x \succsim_{(U_i)_{i\in I}} y$ by Lemma 1. Hence F satisfies Independence. Finally, if F only satisfies CF then all the arguments in this paragraph remain valid provided that $z \in \hat{X}_{(U_i)_{i\in I}}$, so that F satisfies Egalitarian Independence.

Conversely, assume that F satisfies Independence. Let $(U_i)_{i\in I}, (U'_i)_{i\in I} \in \mathcal{P}^I$ such that there exist $a \in \mathbb{R}_{++}$ and $(b_i : U_i \to \mathbb{R})_{i\in I}$ such that $U'_i = \{au_i + b_i(u_i) : u_i \in U_i\}$ for all $i \in I$. Let $x, y \in X$ be distinct. Since the affine dimension of X is at least 2, there exists $z \in X$ such that (x, y, z) are affinely independent. For all $i \in I$ and all $u_i \in U_i$, define $v_{i,u_i} \in P$ by

$$v_{i,u_i}(x) = u_i(x), \qquad v_{i,u_i}(y) = u_i(y), \qquad v_{i,u_i}(z) = \frac{b_i(u_i)}{a}, \qquad v_i(w) = 0 \text{ for all } w \in Y \setminus \{x, y, z\}.$$

Let $V_i = \{v_{i,u_i} : u_i \in U_i\} \in \mathcal{P}$. It suffices to show that $x \succsim_{(U_i)_{i\in I}} y$ if and only if $x \succsim_{(U'_i)_{i\in I}} y$. We first claim that this equivalence holds in the special case where $b_i(u_i) = 0$ for all $i \in I$ and all $u_i \in U_i$. If

$a = 1$ then the claim is trivial since $U_i = U_i'$ for all $i \in I$. If $a \neq 1$ then, swapping U_i and U_i' if needed, we can assume without loss of generality that $a < 1$. Then

$$x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow x \succsim_{(V_i)_{i\in I}} y \Leftrightarrow ax + (1-a)z \succsim_{(V_i)_{i\in I}} ay + (1-a)z \Leftrightarrow x \succsim_{(U_i')_{i\in I}} y,$$

where the first and third equivalences follow from Lemma 1 and the second one from Independence. Now for the general case, let $V_i' = \{0.5u_i + 0.5b_i(u_i)/a \, : \, u_i \in U_i\} = \{u_i'/2a \, : \, u_i' \in U_i'\} \in \mathcal{P}$ for all $i \in I$. Then

$$x \succsim_{(U_i)_{i\in I}} y \Leftrightarrow x \succsim_{(V_i)_{i\in I}} y \Leftrightarrow 0.5x + 0.5z \succsim_{(V_i)_{i\in I}} 0.5y + 0.5z \Leftrightarrow x \succsim_{(V_i')_{i\in I}} y \Leftrightarrow x \succsim_{(V_i)_{i\in I}} y,$$

where the first and third equivalences follow from Lemma 1, the second one from Independence, and the fourth one from the above claim. Hence F satisfies CU. Finally, if F only satisfies Egalitarian Independence then all the arguments in this paragraph remain valid provided that $b_i(u_i) = b_j(u_j)$ for all $i, j \in I$ and all $u_i \in U_i, u_j \in U_j$, so that F satisfies CF. □

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